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Magnetoplasma Wave Properties

by

Paul Diament

February 1967

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INSTITUTE FOR PLASMA RESEARCH
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MAGNETOPLASMA WAVE PROPERTIES

by

Paul Diament

Technical Report

U.S. Army Electronics Command
Fort Monmouth, New Jersey

Contract DA-28-043-AMC-02041(E)
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Institute for Plasma Research
Stanford University
Stanford, California

ABSTRACT

A method is presented for unifying the analysis of various wave properties of a plasma in a magnetic field. An expression is derived for any microscopic perturbation quantity as an integral of an expectation value with respect to the equilibrium distribution. This yields permittivity and conductivity tensors, and hence the dispersion relation, or spatial and temporal decay or growth rates, for any specified velocity distribution. In the plane wave case, the averaging is eliminated and the calculation significantly simplified by transformation to "inverse velocity space," so that singular integrals do not appear and phenomena such as Landau damping become evident. Quasistatic and exact dispersion relations are derived for cold, Maxwellian, resonance, and drifting distributions. Collisions are accounted for as a viscous drag force along the orbits. Generalizations to other external force fields are discussed.

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PRINCIPAL SYMBOLS

$f(\underline{r}, \underline{v}, t)$	Distribution function in velocity space.
$F(\underline{r}, \underline{\Lambda}, t)$	Distribution function in inverse velocity space.
$\underline{\Lambda}$	Position vector in inverse velocity space.
$\underline{a}(\underline{r}, \underline{v}, t)$	Acceleration suffered by plasma constituent at point $\underline{r}$ with velocity $\underline{v}$ at time t.
$\hat{\underline{b}}$	Unit vector along uniform external magnetic field $\underline{B}_0$.
ω_c	$(q/m)B_0$, signed cyclotron frequency.
ν	Effective collisional relaxation rate.
$\underline{I}$	Unit matrix.
$\underline{\parallel}$	$\hat{\underline{b}}\hat{\underline{b}}$, projection matrix, along magnetic field.
$\underline{\perp}$	$\underline{I} - \hat{\underline{b}}\hat{\underline{b}}$, projection matrix, perpendicular to magnetic field.
$\underline{X}$	$\hat{\underline{b}} \times$, matrix representation of cross product with $\hat{\underline{b}}$.
$\underline{R}$	$\frac{1}{2}(\underline{\perp} - i\underline{X})$, right circular polarization matrix.
$\underline{L}$	$\frac{1}{2}(\underline{\perp} + i\underline{X})$, left circular polarization matrix.
$\underline{Y}$	$\nu\underline{I} - \omega_c\underline{X}$, coefficient matrix in orbit equation.
$\underline{v}_0$	Mean flow velocity of unperturbed distribution.
T	Temperature, in energy units.
w	$t' - t$, elapsed time along trajectory.
$\underline{v}'$	$\underline{v}_0 + e^{-\underline{Y}w}(\underline{v} - \underline{v}_0)$, velocity at time t' on orbit.
$\underline{z}$	$\underline{v}_0 w + (\underline{I} - e^{-\underline{Y}w})\underline{Y}^{-1}(\underline{v} - \underline{v}_0)$, displacement from present position on orbit.
$\underline{U}(w)$	$e^{-\underline{Y}(t'-t)}$, velocity transfer function along orbit.
$\underline{S}(w)$	$(\underline{I} - e^{-\underline{Y}(t'-t)})\underline{Y}^{-1}$, position transfer function along orbit.

α_t , α_s	Temporal and spatial decay rates for field amplitudes.
$\underline{A}'$	Transpose of matrix $\underline{A}$.
eig $\underline{A}$	An eigenvalue of matrix $\underline{A}$.
$E(z)$	$e^{z^2} \int_z^\infty e^{-t^2} dt$, error function, with complex argument.

1. INTRODUCTION

The basis of a tractable analysis of wave dispersion, stability, amplification, and interaction in a magnetized plasma is to be found in a self-consistent perturbation calculation of the fields associated with a given equilibrium velocity distribution of the plasma constituents. The underlying hypothesis is that the waves represent disturbances which are only small perturbations of the system. The key to the analysis is the determination of the current (or charge) resulting from volume forces in the plasma, together with the force fields produced by such currents. To that end, the calculation combines Maxwell's equations, or their equivalent, with a linearized version of the Boltzmann equation.

The solution of the linearized Boltzmann equation has become a well-established procedure.[1-6] The analysis presented herein distinguishes itself in the following respects, all contributing to the goal of keeping the theory general, yet tractable.

Extensive use is made of matrix calculus, particularly to simplify the description of the orbit of a particle in the external magnetic field. Collisions are accounted for, phenomenologically, as an additional viscous retarding force affecting the unperturbed orbit. The velocity distributions are kept three dimensional throughout. The quantities to be evaluated are expressed directly as expectation values with respect to the unperturbed velocity distribution, rather than through the velocity gradients. Various functionals of the current density are readily accommodated in the calculation, so that energy

and spatial decay or growth rates are obtainable directly. The quasi-static approximation, although often convenient, is not necessary to the analysis and both quasistatic and exact results are presented.

Singular integrals, requiring careful specification of the integration contours in the complex plane, are avoided, leaving straightforward quadratures that are either standard integrals or types readily handled by computers. Landau and cyclotron damping effects appear in a natural, unforced manner, without complex contour integration or calculation of residues at singularities.

Plane wave or Fourier analysis is performed in a particularly simple manner by transformation of the velocity distribution to "inverse velocity space," in which significant algebraic simplification is achieved. The effects of drift or beams become readily determinable. The approach is easily generalized to systems externally forced otherwise than by the magnetic field, hence may incorporate parametric effects.

2. SOLUTION OF BOLTZMANN EQUATION

The Boltzmann equation governing the velocity distribution f of a constituent of the plasma may be expressed as

$$\frac{\partial f}{\partial t} + \underset{\sim}{v} \cdot \nabla f + \underset{\sim}{a} \cdot \frac{\partial f}{\partial \underset{\sim}{v}} = \left(\frac{\partial f}{\partial t}\right)_c \quad . \tag{2.1}$$

Although it is standard[6] to separate the collisional rate of change as a forcing term, the effect of collisions will here be incorporated in the acceleration, a , experienced by the plasma constituent, as detailed in Section 4.

The nonlinear equation will be simplified at the outset by linearizing about an equilibrium distribution $f_0(\underset{\sim}{v})$. Thus, the distribution function will be

$$f = f_0(\underset{\sim}{v}) + f_1(\underset{\sim}{r}, \underset{\sim}{v}, t) \quad , \tag{2.2}$$

where f_0 conforms to the externally imposed force field producing the acceleration $\underset{\sim}{a}_0$ in the constituent particles, and the perturbation f_1 is associated with the internally induced fields producing particle acceleration $\underset{\sim}{a}_1$. With some additional effort, the more general case of a spatially varying equilibrium distribution $f_0(\underset{\sim}{r}, \underset{\sim}{v})$ could be treated as well.

Linearization prescribes

$$\underset{\sim}{a}_0 \cdot \frac{\partial f_0}{\partial \underset{\sim}{v}} = 0 \tag{2.3}$$

as a precondition for equilibrium, and

$$\frac{\partial f_1}{\partial t} + \underset{\sim}{v} \cdot \nabla f_1 + \underset{\sim}{a}_0 \cdot \frac{\partial f_1}{\partial \underset{\sim}{v}} = -\underset{\sim}{a}_1 \cdot \frac{\partial f_0}{\partial \underset{\sim}{v}} \qquad (2.4)$$

as the equation governing the perturbation of the distribution. How the collision term is to enter these two equations is discussed in Section 4.

In these equations, $\underset{\sim}{a}_0(\underset{\sim}{r}, \underset{\sim}{v}, t)$ is a prescribed external excitation and the equilibrium distribution $f_0(\underset{\sim}{v})$ is presumed known. The first-order acceleration $\underset{\sim}{a}_1(\underset{\sim}{r}, \underset{\sim}{v}, t)$ is specified in terms of the r.f. fields in the plasma, which in turn depend on the unknown perturbation f_1. A self-consistent solution of this equation is required in order to determine the wave properties of the medium.

The solution to the linearized Boltzmann equation is obtained by integrating along the unperturbed trajectory of a constituent particle experiencing the acceleration $\underset{\sim}{a}_0$. Provided that

$$\frac{d\underset{\sim}{r}}{dt} = \underset{\sim}{v} \quad , \quad \frac{d\underset{\sim}{v}}{dt} = \underset{\sim}{a}_0 \quad , \qquad (2.5)$$

the equation states that

$$\frac{df_1}{dt} = -\underset{\sim}{a}_1 \cdot \frac{\partial f_0}{\partial \underset{\sim}{v}} \quad , \qquad (2.6)$$

or simply the total time derivative along the trajectory specified by (2.5). The solution which vanishes at $t = -\infty$, before the perturbation sets in, is

$$f_1(\underset{\sim}{r}, \underset{\sim}{v}, t) = -\int_{-\infty}^{t} \frac{\partial f_0}{\partial \underset{\sim}{v}} \cdot \underset{\sim}{a}_1\left(\underset{\sim}{r}(\tau), \underset{\sim}{v}(\tau), \tau\right) d\tau \quad . \tag{2.7}$$

The integrand is evaluated along the orbit given by (2.5). As this orbit is to pass through $\underset{\sim}{r}$ with velocity $\underset{\sim}{v}$ at the time t, the integral depends on $\underset{\sim}{r}$ and $\underset{\sim}{v}$ as well as t. It is implicit in (2.7) that the perturbation has a starting time. A steady-state analysis can be reconciled with this by tacitly including a small loss component,[7] or an adiabatic switching factor,[8] to guarantee convergence of the improper integral.

3. PERTURBATION QUANTITIES

The various quantities of physical interest associated with the plasma are expressible as expectation values of certain functions of velocity. Quantities that vanish at equilibrium are obtainable by averaging with just the perturbation $f_1(\underset{\sim}{r}, \underset{\sim}{v}, t)$, typically yielding the perturbation quantity $\Phi_1(\underset{\sim}{r}, t)$ as the average of a function $\phi(\underset{\sim}{v})$:

$$\Phi_1(\underset{\sim}{r}, t) = \int f_1(\underset{\sim}{r}, \underset{\sim}{v}, t)\, \phi(\underset{\sim}{v})\, d\underset{\sim}{v} \quad . \tag{3.1}$$

The excess charge $\rho(\underset{\sim}{r}, t)$ is obtainable in this way with $\phi(\underset{\sim}{v}) = q$ and the r.f. current density $\underset{\sim}{J}(\underset{\sim}{r}, t)$ with $\phi(\underset{\sim}{v}) = q\underset{\sim}{v}$. More generally, $\phi(\underset{\sim}{v})$ may represent a velocity-dependent operator acting on the coordinates.

In view of (2.7), any such quantity may be calculated as

$$\Phi_1(\underset{\sim}{r}, t) = -\int \int_{-\infty}^{t} \frac{\partial f_0}{\partial \underset{\sim}{v}} \cdot \underset{\sim}{a}_1\, \phi(\underset{\sim}{v})\, d\tau\, d\underset{\sim}{v} \quad . \tag{3.2}$$

This expression may be simplified by invoking the divergence theorem in velocity space, along with the condition that $f_0 \to 0$, strongly, as $|\underset{\sim}{v}| \to \infty$, leaving

$$\Phi_1(\underset{\sim}{r}, t) = \int_{-\infty}^{t} \int f_0(\underset{\sim}{v})\, \frac{\partial}{\partial \underset{\sim}{v}} \cdot \left[\frac{\partial \underset{\sim}{v}(t)}{\partial \underset{\sim}{v}(\tau)} \cdot \underset{\sim}{a}_1\, \phi(\underset{\sim}{v}) \right] d\underset{\sim}{v}\, d\tau \quad . \tag{3.3}$$

The tensor factor in the brackets arises from the fact that, in (3.2), $\partial f_0/\partial \underset{\sim}{v}$ is evaluated at $\underset{\sim}{v}(\tau)$ while the velocity integration is with respect to the variable $\underset{\sim}{v} = \underset{\sim}{v}(t)$.

Using angular brackets to denote averages with respect to the unperturbed distribution function, as

$$\int f_0(\underset{\sim}{v})\, d\underset{\sim}{v} = n_0 \;, \qquad n_0\langle g(\underset{\sim}{v})\rangle = \int f_0(\underset{\sim}{v})\, g(\underset{\sim}{v})\, d\underset{\sim}{v} \;, \tag{3.4}$$

the final result is expressible as

$$\phi_1(\underset{\sim}{r}, t) = n_0 \int_{-\infty}^{t} \left\langle \frac{\partial}{\partial \underset{\sim}{v}} \cdot \left[\frac{\partial \underset{\sim}{v}}{\partial \underset{\sim}{v}(\tau)} \cdot \underset{\sim}{a}_1(\tau)\, \phi(\underset{\sim}{v}) \right] \right\rangle d\tau \;. \tag{3.5}$$

This expression is the basis for the calculation of all perturbation quantities of interest. The further development of this formula requires the introduction of the unperturbed trajectory $\underset{\sim}{r}(\tau)$, $\underset{\sim}{v}(\tau)$ as a function of $\underset{\sim}{r}$, $\underset{\sim}{v}$, t .

4. UNPERTURBED ORBIT

The general result may now be specialized to the case of a constant, uniform magnetic field $\underset{\sim}{B}_0$ as the external force field. This introduces a preferred direction in space, that of $\underset{\sim}{B}_0$, to be specified by the unit vector $\hat{\underset{\sim}{b}}$. It is then convenient to define a trio of matrices associated with $\hat{\underset{\sim}{b}}$, as follows.

$$\underset{\sim}{\parallel} = \hat{\underset{\sim}{b}}\hat{\underset{\sim}{b}} , \quad \underset{\sim}{\perp} = \underset{\sim}{I} - \hat{\underset{\sim}{b}}\hat{\underset{\sim}{b}} , \quad \underset{\sim}{X} = \hat{\underset{\sim}{b}} \times . \tag{4.1}$$

Here, $\underset{\sim}{I}$ is the unit matrix; $\underset{\sim}{\parallel}$ and $\underset{\sim}{\perp}$ are seen to be projection operators and $\underset{\sim}{X}$ performs the cross product operation. A mutually orthogonal, idempotent set of matrices is formed by $\underset{\sim}{R}$, $\underset{\sim}{L}$, $\underset{\sim}{\parallel}$, where

$$\underset{\sim}{R} = \frac{1}{2}(\underset{\sim}{\perp} - i\underset{\sim}{X}) , \quad \underset{\sim}{L} = \frac{1}{2}(\underset{\sim}{\perp} + i\underset{\sim}{X}) . \tag{4.2}$$

These represent right- and left-handed circular polarization operators, respectively. The spectral expansion of the operator $\underset{\sim}{X}$ shows that any matrix function of $\underset{\sim}{X}$ reduces to

$$f(\underset{\sim}{X}) = f(i)\,\underset{\sim}{R} + f(-i)\,\underset{\sim}{L} + f(0)\,\underset{\sim}{\parallel} . \tag{4.3}$$

The utility of these definitions arises from the fact that in the external magnetic field $\underset{\sim}{B}_0$, the constituent particle acceleration

$\underset{\sim}{a}_0$ produced is

$$\underset{\sim}{a}_0 = (q/m)\ \underset{\sim}{v} \times \underset{\sim}{B}_0 = -\omega_c\ \underset{\sim}{X}\ \underset{\sim}{v} \quad , \tag{4.4}$$

where $\omega_c = qB_0/m$ is the signed cyclotron frequency. With $\underset{\sim}{a}_0 = d\underset{\sim}{v}/dt$, this equation describes the precession of the velocity vector about the magnetic field.

One consequence of the form of the acceleration imposed by the external magnetic field is that the condition (2.3) on the unperturbed velocity distribution becomes

$$\frac{\partial f_0}{\partial \underset{\sim}{v}} \cdot \underset{\sim}{X} \cdot \underset{\sim}{v} = \hat{\underset{\sim}{b}} \cdot \underset{\sim}{v} \times \frac{\partial f_0}{\partial \underset{\sim}{v}} = \frac{\partial f_0}{\partial \varphi} = 0 \quad , \tag{4.5}$$

which precludes any velocity-space azimuthal (φ) dependence of the equilibrium distribution.

Although the introduction of the orbit into (3.5) is now straightforward for the collisionless case, it is desirable at this point to incorporate the effects of collisions, in some manner. This would at least serve to resolve ambiguities associated with singularities appearing in the absence of collisions. Some of these arise from the fact that the matrix $\underset{\sim}{X}$ is singular. To maintain tractability, collisions are here to be included in the simplest fashion, in terms of an equivalent "collision frequency" or inverse relaxation time, ν . This is commonly introduced in any of various convenient or reasonable approximations[9-11] to the collisional term in (2.1). It should be noted that

a simple relaxation term in the Boltzmann equation is inadequate when the quasistatic approximation is to be used ab initio, due to its failure to conserve particles locally.[9]

The artifice to be employed here to represent collisional effects is to ascribe them to merely a modification of the unperturbed orbit. The typical particle is considered to be subject to a viscous drag force, in addition to the magnetic one, to the extent that its velocity differs from the mean flow velocity at equilibrium. Besides the resultant tractability, this method has the virtues of yielding results consistent with the limiting case of cold plasma hydrodynamic theory, as well as consistency between quasistatic and exact theory. Although this approach neglects diffusion in velocity space, the proposed change in the unperturbed orbit appears also as an essential modification of the collisionless case when more careful account is taken of collisions via a Fokker-Planck model.[11]

Accordingly, the acceleration in (4.4), which prescribes the orbit, is modified to

$$\underset{\sim}{a}_0 = -\underset{\sim}{\omega_c}\underset{\sim}{X}\,\underset{\sim}{v} - \nu\,(\underset{\sim}{v} - \underset{\sim}{v}_0) \quad , \tag{4.6}$$

where ν is the effective collision frequency, assumed constant, and $\underset{\sim}{v}_0 = \langle \underset{\sim}{v} \rangle$ is the mean, d.c. drift velocity of the equilibrium distribution. The acceleration is still linear in the velocity in this model and the orbit is readily expressed, from (2.5), by

$$d\underset{\sim}{r}/d\tau = \underset{\sim}{v}(\tau) \quad , \quad d\underset{\sim}{v}/d\tau = -(\underset{\sim}{\omega_c}\underset{\sim}{X} + \nu)\underset{\sim}{v} + \nu\underset{\sim}{v}_0 \quad , \tag{4.7}$$

subject to the conditions $\underset{\sim}{r}(\tau) = \underset{\sim}{r}$, $\underset{\sim}{v}(\tau) = \underset{\sim}{v}$ at $\tau = t$. Since $\underset{\sim}{v}_0$ is to be time-independent and is necessarily aligned with the magnetic field, the orbit equation actually simplifies to

$$\frac{d}{dt}(\underset{\sim}{v} - \underset{\sim}{v}_0) = -(\omega_o \underset{\sim}{X} + \nu)(\underset{\sim}{v} - \underset{\sim}{v}_0) \quad , \tag{4.8}$$

in terms of the peculiar, or random, velocity $\underset{\sim}{v} - \underset{\sim}{v}_0$ and the nonsingular matrix $\omega_o \underset{\sim}{X} + \nu = \underset{\sim}{Y}$.

The solution for the trajectory is

$$\underset{\sim}{v}(\tau) - \underset{\sim}{v}_0 = e^{-\underset{\sim}{Y}(\tau - t)}(\underset{\sim}{v} - \underset{\sim}{v}_0) \quad ; \tag{4.9}$$

$$\underset{\sim}{r}(\tau) = \underset{\sim}{r} + \underset{\sim}{v}_0(\tau - t) + [1 - e^{-\underset{\sim}{Y}(\tau - t)}]\, \underset{\sim}{Y}^{-1}(\underset{\sim}{v} - \underset{\sim}{v}_0) \quad . \tag{4.10}$$

Equation (4.9) expresses the damped precession of the peculiar velocity vector about $\hat{\underset{\sim}{b}}$, until it attains $\underset{\sim}{v} - \underset{\sim}{v}_0$ at time t ; eq. (4.10) describes the constricted helical path taken by the particle which arrives at $\underset{\sim}{r}$ with velocity $\underset{\sim}{v}$ at time t , as indicated in Fig. 1.

By use of (4.3), these expressions may be rendered more explicit, for reference purposes. Let $\tau = t + u$, $\underset{\sim}{r}(\tau) = \underset{\sim}{r} + \underset{\sim}{s}$, $\underset{\sim}{v}(\tau) = \underset{\sim}{w}$. Then

$$\underset{\sim}{w} = \underset{\sim}{v}_0 + \underset{\sim}{U}(u) \cdot (\underset{\sim}{v} - \underset{\sim}{v}_0) \quad , \tag{4.11}$$

$$\underset{\sim}{s} = \underset{\sim}{v}_0 u + \underset{\sim}{S}(u) \cdot (\underset{\sim}{v} - \underset{\sim}{v}_0) \quad , \tag{4.12}$$

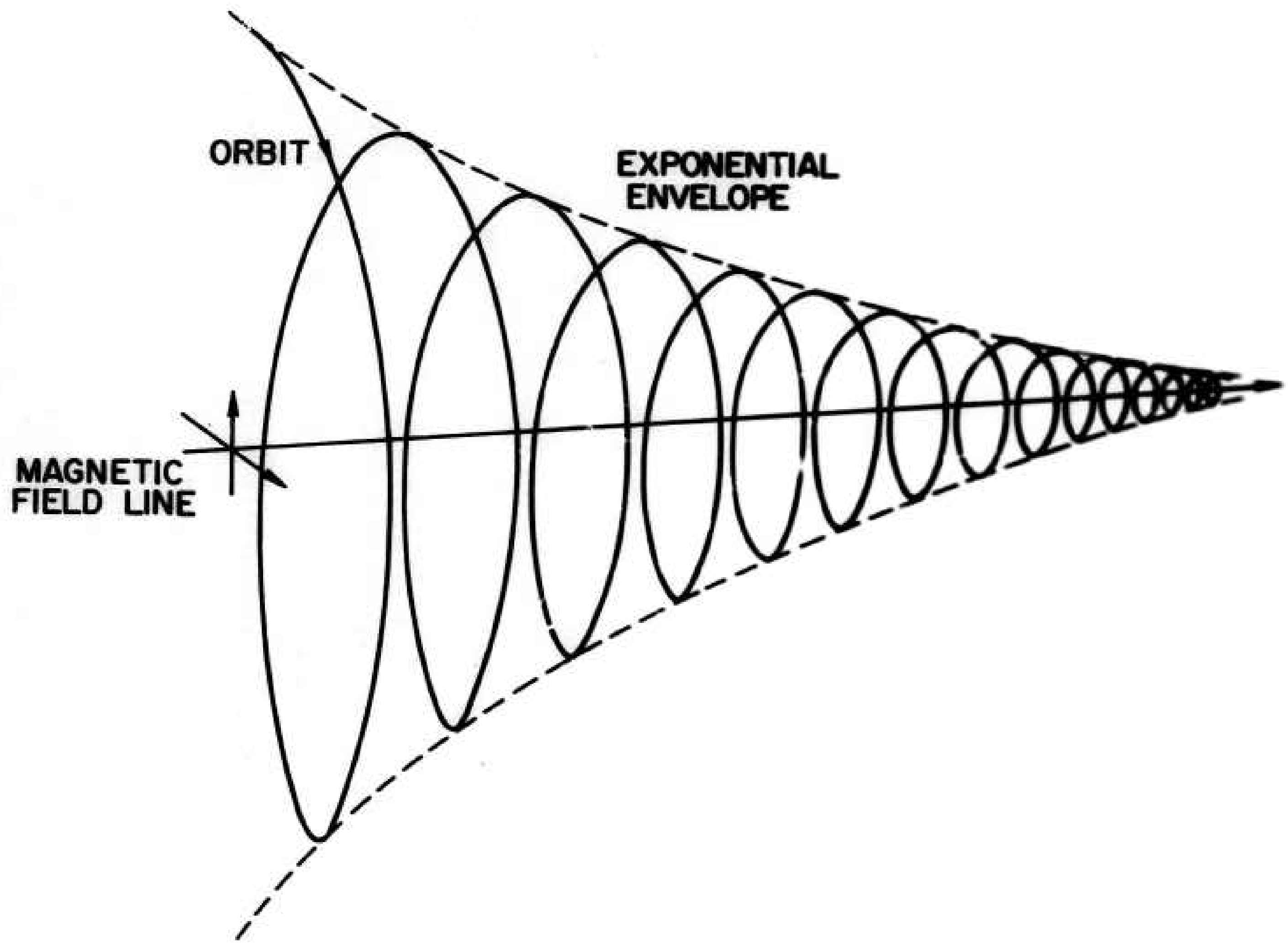

Figure 1 - Damped helical trajectory of "average particle" in magnetoplasma

where

$$\underset{\sim}{V}(u) = e^{-\underset{\sim}{Y}u} = e^{-\nu u}\left(e^{-i\omega_c u}\underset{\sim}{R} + e^{i\omega_c u}\underset{\sim}{L} + \underset{\sim}{\|}\right)$$

$$= e^{-\nu u}\left(\cos(\omega_c u)\underset{\sim}{\perp} - \sin(\omega_c u)\underset{\sim}{X} + \underset{\sim}{\|}\right) \qquad (4.13)$$

and

$$\underset{\sim}{S}(u) = (1 - e^{-\underset{\sim}{Y}u})\,\underset{\sim}{Y}^{-1}$$

$$= \frac{1 - e^{-(\nu + i\omega_c)u}}{\nu + i\omega_c}\underset{\sim}{R} + \frac{1 - e^{-(\nu - i\omega_c)u}}{\nu - i\omega_c}\underset{\sim}{L} + \frac{1 - e^{-\nu u}}{\nu}\underset{\sim}{\|} \quad . \qquad (4.14)$$

Note that, since $\underset{\sim}{v}_0 = \underset{\sim}{\|}\,\underset{\sim}{v}_0$,

$$g(\underset{\sim}{Y})\,\underset{\sim}{v}_0 = g(\nu)\,\underset{\sim}{v}_0 \qquad (4.15)$$

for any matrix function. Also, the result $\underset{\sim}{S}(u)e^{\underset{\sim}{Y}u} = -\underset{\sim}{S}(-u)$ will be useful later.

It may be noted that, by virtue of the time-invariance of the system, the particle dynamics are entirely expressible in terms of the elapsed time $u = \tau - t$. Finally, for use in (3.5), note that $\partial\underset{\sim}{v}/\partial\underset{\sim}{v}(\tau) = e^{\underset{\sim}{Y}u}$.

Substitution of these results in (3.5) makes it applicable to the magnetized plasma:

$$G_1(\mathbf{r}, t) = n_0 \int_{-\infty}^{0} \left\langle \frac{\partial}{\partial \mathbf{v}} \cdot \left[e^{[illegible]} \cdot \mathbf{a}_1(\mathbf{r}', \mathbf{v}', t+\tau) [illegible] \right] \right\rangle d\tau \qquad (4.16)$$

In the integrand, $\mathbf{r}'$ and $\mathbf{v}'$ are functions of $\mathbf{v}$ and of τ , as indicated in (4.11,12). This expression gives the first-order perturbation of the macroscopic quantity corresponding to $g(\mathbf{v})$ due to constituents of charge q and mass m of the magnetized plasma, which undergo acceleration $\mathbf{a}_1(\mathbf{r}, \mathbf{v}, t)$ in the induced force fields. The latter are the electromagnetic fields excited in the plasma, so that

$$\mathbf{a}_1(\mathbf{r}, \mathbf{v}, t) = (q/m) \left[\mathbf{E}_1(\mathbf{r}, t) + \mathbf{v} \times \mathbf{B}_1(\mathbf{r}, t)\right] . \qquad (4.17)$$

The sources of $\mathbf{E}_1$, $\mathbf{B}_1$ are the charge and current densities, obtainable in turn from (4.16).

5. MACROSCOPIC OBSERVABLES

The basic physical quantities of interest are the excess charge and current densities in the plasma. These may be calculated by taking $\phi(\underset{\sim}{v}) = q$ and $q\underset{\sim}{v}$, respectively, in (4.16):

$$\rho(\underset{\sim}{r},\ t) = n_0 q \int_{-\infty}^{0} \left\langle \frac{\partial}{\partial \underset{\sim}{v}} \cdot \left[e^{(\omega_c \underset{\sim}{X} + \nu)u} \cdot \underset{\sim}{a}_1(\underset{\sim}{r}+\underset{\sim}{s},\ \underset{\sim}{w},\ t+u) \right] \right\rangle du \quad ; \tag{5.1}$$

$$\underset{\sim}{J}(\underset{\sim}{r},\ t) = n_0 q \int_{-\infty}^{0} \left\langle \frac{\partial}{\partial \underset{\sim}{v}} \cdot \left[e^{(\omega_c \underset{\sim}{X} + \nu)u} \cdot \underset{\sim}{a}_1(\underset{\sim}{r}+\underset{\sim}{s},\ \underset{\sim}{w},\ t+u)\ \underset{\sim}{v} \right] \right\rangle du \quad . \tag{5.2}$$

It may be noted at this point that the general result of eq. (4.16) reduces to the following expressions in the cases of harmonic time variation and of plane waves. With $\underset{\sim}{a}_1(\underset{\sim}{r},\ \underset{\sim}{v},\ t) = \mathrm{Re}\ [\underset{\sim}{a}_1(\underset{\sim}{r},\underset{\sim}{v})e^{i\omega t}]$ or $\underset{\sim}{a}_1(\underset{\sim}{r},\ \underset{\sim}{v},\ t) = \mathrm{Re}\ [\underset{\sim}{a}_1(\underset{\sim}{v})e^{i(\omega t - \underset{\sim}{k}\cdot\underset{\sim}{r})}]$, and corresponding expressions for the macroscopic quantities of interest,

$$\Phi_1(\underset{\sim}{r}) = n_0 \int_{-\infty}^{0} \left\langle \frac{\partial}{\partial \underset{\sim}{v}} \cdot \left[e^{(\omega_c \underset{\sim}{X} + \nu + i\omega)u} \cdot \underset{\sim}{a}_1(\underset{\sim}{r}+\underset{\sim}{s},\ \underset{\sim}{w})\ \phi(\underset{\sim}{v}) \right] \right\rangle du \quad ; \tag{5.3}$$

$$\Phi_1 = n_0 \int_{-\infty}^{0} \left\langle \frac{\partial}{\partial \underset{\sim}{v}} \cdot \left[e^{(\omega_c \underset{\sim}{X} + \nu + i\omega)u}\ e^{-i\underset{\sim}{k}\cdot\underset{\sim}{s}} \cdot \underset{\sim}{a}_1(\underset{\sim}{w})\ \phi(\underset{\sim}{v}) \right] \right\rangle du \quad . \tag{5.4}$$

Wave properties of the medium are most readily obtained from quantities derived from the charge and current, rather than from the sources themselves. Dispersion relations are expressible in terms of susceptibilities and spatial and temporal decay rates in terms of the

power balance. To obtain this, Maxwell's equations, or their equivalent, must be combined with the above expressions.

The medium may be considered as either dielectric or conducting, as convenient. The dielectric susceptibility $\underset{\sim}{K}(\underset{\sim}{k},\omega)$ is so defined for plane waves that

$$\rho = -i\epsilon_0 \underset{\sim}{k} \cdot \underset{\sim}{K}(\underset{\sim}{k},\omega) \cdot \underset{\sim}{E}_1 \tag{5.5}$$

and the normalized conductivity $\underset{\sim}{C}(\underset{\sim}{k},\omega)$ is defined by

$$\underset{\sim}{J} = -i\epsilon_0 \omega \, \underset{\sim}{C}(\underset{\sim}{k},\omega) \cdot \underset{\sim}{E}_1 \ . \tag{5.6}$$

The dispersion relation for wave propagation is conveniently expressed in terms of these tensors. Under the quasistatic approximation, it suffices to set

$$k^2 = \underset{\sim}{k} \cdot \underset{\sim}{K}(\underset{\sim}{k},\omega) \cdot \underset{\sim}{k} \ , \tag{5.7}$$

since $\underset{\sim}{E}_1$ is considered to be the gradient of a plane wave potential. The exact plane wave dispersion relation, however, is obtained by combining Maxwell's equations into a wave equation, with the current as source. This yields

$$\mathrm{eig}\left[\frac{c^2}{\omega^2}\, \underset{\sim}{k}\,\underset{\sim}{k} - \underset{\sim}{C}(\underset{\sim}{k},\omega)\right] = \left(\frac{ck}{\omega}\right)^2 - 1 \ , \tag{5.8}$$

where the notation $\mathrm{eig}\ \underset{\sim}{A} = \lambda$ means that λ is an eigenvalue of $\underset{\sim}{A}$.

The equivalent permittivity and conductivity tensors are obtainable from (5.4) by translating (4.17) into plane wave notation. The magnetic field may be consistently dropped under the quasistatic assumption, leaving $\underset{\sim}{a}_1 = (q/m)\underset{\sim}{E}_1$, but the exact acceleration is

$$\underset{\sim}{a}_1(\underset{\sim}{v}) = (q/m)\left[(1 - \frac{\underset{\approx}{k}}{\omega}\cdot\underset{\sim}{v})\,\underset{\sim}{I} + \frac{\underset{\approx}{k}}{\omega}\,\underset{\sim}{v}\right]\cdot\underset{\sim}{E}_1 \quad . \qquad (5.9)$$

Substituting in (5.4) and writing $n_0q^2/m\epsilon_0$ as ω_p^2 , the square of the plasma frequency, leads to

$$\underset{\sim}{k}\cdot\underset{\sim}{K}(\underset{\sim}{k},\omega) = i\omega_p^2\int_{-\infty}^{0}\left\langle\frac{\partial}{\partial\underset{\sim}{v}}\cdot\left[e^{(\omega_c\underset{\sim}{X}+\nu+i\omega)u}\,e^{-i\underset{\sim}{k}\cdot\underset{\sim}{s}}\right]\right\rangle du \qquad (5.10)$$

and

$$\underset{\sim}{C}(\underset{\sim}{k},\omega)\cdot\underset{\sim}{E}_1 = i\,\frac{\omega_p^2}{\omega}\int_{-\infty}^{0}\left\langle\frac{\partial}{\partial\underset{\sim}{v}}\cdot\left[e^{(\omega_c\underset{\sim}{X}+\nu+i\omega)u}\,e^{-i\underset{\sim}{k}\cdot\underset{\sim}{s}}\left\{(1 - \frac{\underset{\approx}{k}}{\omega}\cdot\underset{\sim}{w})\underset{\sim}{I} + \frac{\underset{\approx}{k}}{\omega}\,\underset{\sim}{w}\right\}\cdot\underset{\sim}{E}_1\,\underset{\sim}{v}\right]\right\rangle du \quad .$$

$$(5.11)$$

Taking the divergences in velocity as indicated simplifies these to

$$\underset{\sim}{K}(\underset{\sim}{k},\omega) = \omega_p^2\int_{-\infty}^{0}\underset{\sim}{S}(u)\cdot e^{(\omega_c\underset{\sim}{X}+\nu+i\omega)u}\left\langle e^{-i\underset{\sim}{k}\cdot\underset{\sim}{s}}\right\rangle du \qquad (5.12)$$

and

$$\underset{\sim}{C}(\underset{\sim}{k},\omega) = i\,\frac{\omega_p^2}{\omega}\int_{-\infty}^{0}\left\langle(\underset{\sim}{I}+\underset{\sim}{v}\,\frac{\partial}{\partial\underset{\sim}{v}}\cdot)\left\{e^{(\omega_c\underset{\sim}{X}+\nu+i\omega)u}\,e^{-i\underset{\sim}{k}\cdot\underset{\sim}{s}}\left[(1 - \frac{\underset{\approx}{k}}{\omega}\cdot\underset{\sim}{w})\underset{\sim}{I}+\frac{\underset{\approx}{k}}{\omega}\,\underset{\sim}{w}\right]\right\}\right\rangle du \quad .$$

$$(5.13)$$

In bounded systems, the quantities of interest are often decay rates, spatial or temporal, particularly when these can be negative, indicating instability or amplifying capabilities. From the power balance in terms of power and energy densities,

$$\nabla \cdot \underset{\sim}{P} + \frac{\partial W}{\partial t} = -\underset{\sim}{J} \cdot \underset{\sim}{E} \quad , \tag{5.14}$$

there is obtained, by appropriate integration in space and time, the decay rates of energy in a cavity or power in a waveguide, as follows.

$$2\alpha_t = \frac{1}{W_{av} T} \int_T \int_V \underset{\sim}{J} \cdot \underset{\sim}{E} \, dV \, dt \quad ; \tag{5.15}$$

$$2\alpha_s = \frac{1}{P_{av} T} \int_T \int_A \underset{\sim}{J} \cdot \underset{\sim}{E} \, dA \, dt \quad . \tag{5.16}$$

In terms of the formalism developed here, these decay rates may be calculated by taking $\phi(\underset{\sim}{v})$ to be $q\underset{\sim}{v} \cdot \underset{\sim}{E}$, integrated in space and time. The result of using this velocity-dependent operator in (4.16) is

$$2\alpha = n_0 \, q \int_{-\infty}^{0} \left\langle \frac{\partial}{\partial \underset{\sim}{v}} \cdot \left[e^{(\omega_c \underset{\sim}{X} + \nu) u} \cdot \underset{\sim}{R}(\underset{\sim}{s}, \underset{\sim}{w}, u) \cdot \underset{\sim}{v} \right] \right\rangle du \quad , \tag{5.17}$$

where the correlation tensor is

$$\underset{\sim}{R}(\underset{\sim}{s}, \underset{\sim}{w}, u) = \frac{1}{W_{av} T} \int_T \int_V \underset{\sim}{a}_1(\underset{\sim}{r} + \underset{\sim}{s}, \, \underset{\sim}{w}, \, t+u) \, \underset{\sim}{E}(\underset{\sim}{r}, t) \, dV \, dt \tag{5.18}$$

for a cavity, or

$$\underset{\sim}{R}(\underset{\sim}{s}, \underset{\sim}{w}, u) = \frac{1}{P_{av}T} \int_T \int_A \underset{\sim}{a}_1(\underset{\sim}{r}+\underset{\sim}{s}, \underset{\sim}{w}, t+u)\, \underset{\sim}{E}(\underset{\sim}{r}, t)\, dA\, dt \qquad (5.19)$$

for a waveguide. Other macroscopic observables, such as frequency shifts, are similarly obtainable by appropriate choice of the microscopic operator $\phi(\underset{\sim}{v})$.

6. INVERSE VELOCITY SPACE

As is evident from (5.12), the basic quantity to be averaged over velocity space has the form

$$e^{-i\underset{\sim}{k}\cdot\underset{\sim}{s}} = e^{-i\underset{\sim}{k}\cdot\underset{\sim}{v}_0 u}\, e^{i\underset{\sim}{\Lambda}\cdot(\underset{\sim}{v} - \underset{\sim}{v}_0)} , \tag{6.1}$$

where

$$\underset{\sim}{\Lambda} = \underset{\sim}{\Lambda}(u) = -\underset{\sim}{k}\cdot\underset{\approx}{S}(u) . \tag{6.2}$$

Define generally, therefore, the distribution function in "inverse velocity space" by

$$F(\underset{\sim}{\Lambda}) = \langle e^{i\underset{\sim}{\Lambda}\cdot\underset{\sim}{v}}\rangle . \tag{6.3}$$

This is just the velocity-space, three-dimensional Fourier transform of the normalized unperturbed distribution function $f_0(\underset{\sim}{v})/n_0$. The equivalent permittivity tensor in (5.12) is then expressible directly in terms of $F(\underset{\sim}{\Lambda})$, evaluated as in (6.2). The averaging of more complicated functions of velocity will often be advantageously expressible in terms of $F(\underset{\sim}{\Lambda})$ as well, as in

$$\langle \underset{\sim}{v} e^{i\underset{\sim}{\Lambda}\cdot\underset{\sim}{v}}\rangle = -i\,\frac{\partial F}{\partial \underset{\sim}{\Lambda}} ; \qquad \langle \underset{\approx}{vv} e^{i\underset{\sim}{\Lambda}\cdot\underset{\sim}{v}}\rangle = -\frac{\partial^2 F}{\partial\underset{\sim}{\Lambda}\partial\underset{\sim}{\Lambda}} . \tag{6.4}$$

In particular, the various velocity moments are easily obtained by evaluating the derivatives of $F(\underset{\sim}{\Lambda})$ at the origin $\underset{\sim}{\Lambda} = 0$; the Laplacian of $F(\underset{\sim}{\Lambda})$ at the origin is $-\langle v^2 \rangle$.

The algebraic form of the distribution function is generally simpler in inverse velocity space, because of the normalization $F(0) = 1$ and the replacement of convolutions by products. In particular, a drifting distribution appears simply as the stationary one multiplied by the exponential factor $e^{i\underset{\sim}{\Lambda}\cdot\underset{\sim}{v}_0}$. Table I gives the form of various distributions in both velocity spaces. Note that the temperature T is taken in energy units.

In terms of the distribution function in inverse velocity space, the equivalent permittivity tensor becomes

$$\underset{\sim}{\underset{\sim}{K}}(\underset{\sim}{k},\omega) = \omega_p^{\,2} \int_{-\infty}^{0} \underset{\sim}{\underset{\sim}{S}}(u)\, e^{[\omega_c \underset{\sim}{\underset{\sim}{X}}+\nu+i(\omega-\underset{\sim}{k}\cdot\underset{\sim}{v}_0)]u}\, e^{-i\underset{\sim}{\Lambda}\cdot\underset{\sim}{v}_0}\, F(\underset{\sim}{\Lambda})du \quad , \quad (6.5)$$

where $\underset{\sim}{\Lambda}(u) = -\underset{\sim}{k}\cdot\underset{\sim}{\underset{\sim}{S}}(u)$ and $\omega-\underset{\sim}{k}\cdot\underset{\sim}{v}_0$ is the Doppler-shifted frequency.

The equivalent conductivity tensor becomes, after some manipulation of (5.13),

$$\underset{\sim}{\underset{\sim}{C}}(\underset{\sim}{k},\omega) = \underset{\sim}{\underset{\sim}{C}}_e(\underset{\sim}{k},\omega) + \underset{\sim}{\underset{\sim}{C}}_b(\underset{\sim}{k},\omega) \quad , \quad (6.6)$$

where

$$\underset{\sim}{\underset{\sim}{C}}_e = i\,\frac{\omega_p^2}{\omega} \int_{-\infty}^{0} \left(F + \frac{\partial F}{\partial \underset{\sim}{\Lambda}}\,\underset{\sim}{\Lambda} \right) \cdot e^{[\omega_c \underset{\sim}{\underset{\sim}{X}}+\nu+i(\omega-\underset{\sim}{k}\cdot\underset{\sim}{v}_0)]u}\, e^{-i\underset{\sim}{\Lambda}\cdot\underset{\sim}{v}_0}\, du \quad (6.7)$$

TABLE I

Distributions in Inverse Velocity Space

1. **General:**

$$f(\mathbf{v}) \qquad\qquad F(\boldsymbol{\Lambda}) = \int f(\mathbf{v}) e^{i\boldsymbol{\Lambda}\cdot\mathbf{v}}\, d^3\mathbf{v} \Big/ \int f(\mathbf{v})\, d^3v$$

2. **Cold:**

$$f(\mathbf{v}) = n_0\, \delta(\mathbf{v}) \qquad\qquad F(\boldsymbol{\Lambda}) = 1$$

3. **Drifting:**

$$f(\mathbf{v}) = f_0(\mathbf{v} - \mathbf{v}_0) \qquad\qquad F(\boldsymbol{\Lambda}) = e^{i\boldsymbol{\Lambda}\cdot\mathbf{v}_0}\, F_0(\boldsymbol{\Lambda})$$

4. **Maxwellian:**

$$f(\mathbf{v}) = n_0 (m/2\pi T)^{3/2}\, e^{-\frac{mv^2}{2T}} \qquad\qquad F(\boldsymbol{\Lambda}) = e^{-\frac{1}{2}\frac{T}{m}\Lambda^2}$$

5. **Resonance:**

$$f(\mathbf{v}) = \frac{n_0}{\pi^2}\, \frac{v_1}{(v^2 + v_1^2)^2} \qquad\qquad F(\boldsymbol{\Lambda}) = e^{-|\boldsymbol{\Lambda}| v_1}$$

6. **Radially symmetric:**

$$f(\mathbf{v}) = f(|\mathbf{v}|) \qquad\qquad F(\boldsymbol{\Lambda}) = \left\langle \frac{\sin|\boldsymbol{\Lambda}||\mathbf{v}|}{|\boldsymbol{\Lambda}|\,|\mathbf{v}|} \right\rangle$$

7. **Cylindrically symmetric:**

$$f(\mathbf{v}) = f(v_\rho) \qquad\qquad F(\boldsymbol{\Lambda}) = \left\langle J_0(\Lambda_\rho v_\rho) \right\rangle$$

8. **Isotropic monoenergetic:**

$$f(\mathbf{v}) = \frac{n_0 \delta(|\mathbf{v}| - v_1)}{4\pi v^2} \qquad\qquad F(\boldsymbol{\Lambda}) = \frac{\sin|\boldsymbol{\Lambda}|\, v_1}{|\boldsymbol{\Lambda}|\, v_1}$$

9. **Transverse monoenergetic:**

$$f(\mathbf{v}) = \frac{n_0 \delta(v_\rho - v_1)\delta(v_z)}{2\pi v_\rho} \qquad\qquad F(\boldsymbol{\Lambda}) = J_0(\Lambda_\rho v_1)$$

10. **Beam in plasma:**

$$f(\mathbf{v}) = n_p f_p(\mathbf{v}) + n_b f_b(\mathbf{v} - \mathbf{v}_0) \qquad\qquad F(\boldsymbol{\Lambda}) = \frac{n_0 - n_b}{n_0} F_p(\boldsymbol{\Lambda}) + \frac{n_b}{n_0} e^{i\boldsymbol{\Lambda}\cdot\mathbf{v}_0} F_b(\boldsymbol{\Lambda})$$

11. **General averages**

$$\langle \Psi(\mathbf{v}) \rangle = \int f(\mathbf{v})\, \Psi(\mathbf{v})\, d^3\mathbf{v}/n_0 = \int F(\boldsymbol{\Lambda})\, \psi(\boldsymbol{\Lambda})\, d^3\boldsymbol{\Lambda}$$

where $\psi(\boldsymbol{\Lambda}) = (2\pi)^{-3} \int \Psi(\mathbf{v})\, e^{-i\boldsymbol{\Lambda}\cdot\mathbf{v}}\, d^3\mathbf{v}$, $\qquad \Psi(\mathbf{v}) = \int \psi(\boldsymbol{\Lambda})\, e^{i\boldsymbol{\Lambda}\cdot\mathbf{v}}\, d^3\boldsymbol{\Lambda}$.

is the contribution of the r.f. electric field, and

$$\underset{\sim}{C}_b = \frac{\omega_p^2}{\omega^2} \underset{\sim}{k} \times \int_{-\infty}^{0} e^{i(\omega - \underset{\sim}{k}\cdot\underset{\sim}{v}_0)u} e^{-i\underset{\sim}{\Lambda}\cdot\underset{\sim}{v}_0} \frac{\partial}{\partial \underset{\sim}{\Lambda}} \left(e^{-\frac{\omega}{c} \underset{\sim}{X} u} \cdot \frac{\partial F}{\partial \underset{\sim}{\Lambda}} \times \underset{\sim}{\Lambda} \cdot e^{\frac{\omega}{c} \underset{\sim}{X} u} \right) du \qquad (6.8)$$

is due to the r.f. magnetic field.

If cavity or waveguide modes are expanded in plane waves, similar reductions of eq. (5.17) may be achieved. There remains to substitute any appropriate distribution function $F(\underset{\sim}{\Lambda})$ into these expressions to yield the dispersion relations or decay rates by straightforward quadrature. Singular integrals do not appear in this formulation.

More generally, the perturbation of the distribution function in inverse velocity space, $F_1(\underset{\sim}{\Lambda})$, is obtainable by setting $n_0\phi(\underset{\sim}{v}) = e^{i\underset{\sim}{\Lambda}\cdot\underset{\sim}{v}}$ in (5.4). Thus, the perturbed distribution is

$$F(\underset{\sim}{r},\underset{\sim}{\Lambda},t) = F(\underset{\sim}{\Lambda}) + F_1(\underset{\sim}{\Lambda}) e^{i(\omega t - \underset{\sim}{k}\cdot\underset{\sim}{r})} , \qquad (6.9)$$

with

$$F_1(\underset{\sim}{\Lambda}) = \int_{-\infty}^{0} \left\langle \frac{\partial}{\partial \underset{\sim}{v}} \cdot \left[e^{\underset{\sim}{Y} u} \cdot \underset{\sim}{a}_1(\underset{\sim}{w}) e^{i(\omega u - \underset{\sim}{k}\cdot\underset{\sim}{s} + \underset{\sim}{\Lambda}\cdot\underset{\sim}{v})} \right] \right\rangle du . \qquad (6.10)$$

The perturbations in charge, current, temperature, etc. are obtainable by evaluating this, and its derivatives, at $\underset{\sim}{\Lambda} = 0$.

7. QUASISTATIC DISPERSION - B = 0

Under the quasistatic approximation, the significant quantity is the equivalent permittivity tensor, given by

$$\underset{\sim}{K}(\underset{\sim}{k},\omega) = \omega_p^2 \int_{-\infty}^{0} \underset{\sim}{S}(u)\, e^{[\omega_c \underset{\sim}{X}+\nu+i(\omega-\underset{\sim}{k}\cdot\underset{\sim}{v}_0)]u}\, e^{-i\underset{\sim}{\Lambda}\cdot\underset{\sim}{v}_0}\, F(\underset{\sim}{\Lambda})\, du \quad . \qquad (7.1)$$

The dispersion relation is then $k^2 = \underset{\sim}{k}\cdot\underset{\sim}{K}(\underset{\sim}{k},\omega)\cdot\underset{\sim}{k}$.

The case of $B_0 = 0$ is recovered in the limit $\omega_c = 0$, whereupon $\underset{\sim}{S}(u) = [(1 - e^{-\nu u})/\nu]\underset{\sim}{I}$, or $\underset{\sim}{S}(u) = u\underset{\sim}{I}$ in the collisionless case, and the permittivity reduces to a scalar: $\underset{\sim}{K}(\underset{\sim}{k},\omega) = \chi(\underset{\sim}{k},\omega)\,\underset{\sim}{I}$.

$$\chi(\underset{\sim}{k},\omega) = \omega_p^2 \int_{-\infty}^{0} \frac{1 - e^{-\nu u}}{\nu}\, e^{[\nu+i(\omega-\underset{\sim}{k}\cdot\underset{\sim}{v}_0)]u}\, e^{-i\underset{\sim}{\Lambda}\cdot v_0}\, F(\underset{\sim}{\Lambda})\, du \quad , \qquad (7.2)$$

where $\underset{\sim}{\Lambda} = (\underset{\sim}{k}/\nu)(e^{-\nu u} - 1)$. The dispersion relation is simply $\chi(\underset{\sim}{k},\omega) = 1$. Note that $\underset{\sim}{v}_0 = -i\, \partial F(0)/\partial\underset{\sim}{\Lambda}$.

a) Cold plasma: $F(\underset{\sim}{\Lambda}) = 1$.

$$\chi(\underset{\sim}{k},\omega) = \omega_p^2 \int_{-\infty}^{0} \frac{1 - e^{-\nu u}}{\nu}\, e^{(\nu+i\omega)u}\, du = \frac{\omega_p^2}{\omega(\omega-i\nu)} \quad . \qquad (7.3)$$

A small loss component has been invoked to make the integral convergent. The dispersion relation here represents merely a damped oscillation at the plasma frequency.

b) Maxwellian distribution: $F(\underset{\sim}{\Lambda}) = e^{-\frac{1}{2}\frac{T}{m}\Lambda^2}$.

Here, $\Lambda^2 = (k/\nu)^2(e^{-\nu u} - 1)^2$ or $\Lambda^2 = k^2u^2$ for $\nu = 0$. The collisionless case is the most tractable, reducing to

$$\chi(k,\omega) = \omega_p^2 \int_{-\infty}^{0} u\, e^{i\omega u}\, e^{-\frac{1}{2}\frac{k^2 T}{m} u^2}\, du$$

$$= \frac{\omega_p^2 m}{k^2 T} \int_{-\infty}^{0} 2\theta\, e^{2y\theta}\, e^{-\theta^2}\, d\theta$$

$$= \frac{\omega_p^2 m}{k^2 T} \frac{d\Xi}{dy} \qquad (7.4)$$

where $y = (i\omega/k)(m/2T)^{1/2}$ and $\Xi(y) = \int_{-\infty}^{0} e^{-\xi^2+2y\xi}\, d\xi$. The integral is the error function, with complex argument. Landau damping,[12,13] is implicit in this result. It appears again more explicitly in the following case.

c) Resonance distribution: $F(\underset{\sim}{\Lambda}) = e^{-v_1|\underset{\sim}{\Lambda}|}$.

Here, $|\underset{\sim}{\Lambda}| = (k/\nu)(e^{-\nu u} - 1)$ or $|\underset{\sim}{\Lambda}| = -ku$ for $\nu = 0$. The collisionless case exhibits damping:

$$\chi(\underset{\sim}{k},\omega) = \omega_p^2 \int_{-\infty}^{0} u e^{i\omega u}\, e^{v_1 k u}\, du = \frac{\omega_p^2}{(\omega - i v_1 k)^2} \quad . \qquad (7.5)$$

Collisionless Landau damping at the rate $v_1 k$ is evident from the dispersion relation. The physical interpretation of this case is obscure, however, as this distribution has no well-defined moments.

d) Beams: $F(\underset{\sim}{\Lambda}) = e^{i\underset{\sim}{\Lambda}\cdot\underset{\sim}{v}_0} F_0(\underset{\sim}{\Lambda})$.

A simple, distributed beam results in only a Doppler shift:

$$\chi(\underset{\sim}{k},\omega) = \chi_0(\underset{\sim}{k},\omega - \underset{\sim}{k}\cdot\underset{\sim}{v}_0) \quad , \qquad (7.6)$$

as is evident from (7.2). For a beam in a stationary plasma, however, $n_0 = n_p + n_b$ and ω_p^2 in (7.2) is to be replaced by $\omega_0^2 = \omega_p^2 + \omega_b^2$. Then $\underset{\sim}{v}_0 = \underset{\sim}{v}_b(n_b/n_0)$ and, in (7.2),

$$\omega_0^2 \, F(\underset{\sim}{\Lambda}) = \omega_p^2 \, F_p \, (\underset{\sim}{\Lambda}) + \omega_b^2 \, e^{i\underset{\sim}{\Lambda} \cdot \underset{\sim}{v}_b} \, F_b(\underset{\sim}{\Lambda}) \quad . \tag{7.7}$$

For equal, opposed, interpenetrating beams, $\underset{\sim}{v}_0 = 0$ and

$$\omega_p^2 \, F(\underset{\sim}{\Lambda}) = 2\omega_b^2 \, \cos \underset{\sim}{\Lambda} \cdot \underset{\sim}{v}_b \, F_b(\underset{\sim}{\Lambda}) \quad . \tag{7.8}$$

8. QUASISTATIC DISPERSION - $B \neq 0$

In a magnetic field, $\underset{\sim}{S}(u)$ is given by (4.14). The product $\underset{\sim}{S}(u)e^{\underset{\sim}{Y}u}$ appearing in (7.1) reduces to $-\underset{\sim}{S}(-u)$, leaving

$$\underset{\sim}{K}(\underset{\sim}{k},\omega) = \omega_p^2 \, \underset{\sim}{Y}^{-1} \int_{-\infty}^{0} (e^{\underset{\sim}{Y}u} - 1)\, e^{i(\omega-\underset{\sim}{k}\cdot\underset{\sim}{v}_0)u}\, e^{i\underset{\sim}{k}\cdot\underset{\sim}{S}(u)\cdot\underset{\sim}{v}_0}\, F\left(-\underset{\sim}{k}\cdot\underset{\sim}{S}(u)\right) du \quad (8.1)$$

where $\underset{\sim}{Y} = \nu + \omega_o\underset{\sim}{X}$. In the collisionless limit, $-\underset{\sim}{S}(-u)$ reduces to

$$-\underset{\sim}{S}(-u) = \frac{\sin\omega_c u}{\omega_c} \underset{\approx}{\perp} + \left(\frac{1 - \cos\omega_c u}{\omega_c}\right)\underset{\sim}{X} + u\underset{\approx}{\parallel} \quad (8.2)$$

and the dispersion relation, which involves

$$-\underset{\sim}{k}\cdot\underset{\sim}{S}(-u)\cdot\underset{\sim}{k} = \frac{\sin\omega_c u}{\omega_c} k_\perp^2 + u\, k_\parallel^2 , \quad (8.3)$$

reduces to

$$k_\perp^2\left(1 - \frac{\omega_p^2}{\omega_c^2} g_\perp\right) + k_\parallel^2\left(1 - \frac{\omega_p^2}{\omega_c^2} g_\parallel\right) = 0 , \quad (8.4)$$

where

$$g_\perp = \int_{-\infty}^{0} \sin\theta \, e^{i\frac{\omega}{\omega_c}\theta} F(\underset{\sim}{\Lambda})\, d\theta \quad (8.5)$$

and

$$g_{\parallel} = \int_{-\infty}^{0} \theta \, e^{i \frac{\omega}{\omega_c} \theta} F(\underset{\sim}{\Lambda}) \, d\theta \quad , \tag{8.6}$$

with $\underset{\sim}{\Lambda} = -(\underset{\sim}{k}/\omega_c) \cdot \left(\sin \theta \underset{\sim}{\perp} - (1 - \cos\theta) \underset{\sim}{X} + \theta \underset{\sim}{\parallel}\right)$.

a) Cold plasma: $F(\underset{\sim}{\Lambda}) = 1$.

From (8.1),

$$\underset{\sim}{K}(\underset{\sim}{k},\omega) = \omega_p^2 \, \underset{\sim}{Y}^{-1} \int_{-\infty}^{0} \left(e^{\underset{\sim}{Y}u} - 1\right) e^{i\omega u} \, du$$

$$= \frac{\omega_p^2}{\omega} (\omega - i\underset{\sim}{Y})^{-1} = \frac{\omega_p^2}{\omega} \left[\frac{\underset{\sim}{R}}{\omega - i\nu + \omega_c} + \frac{\underset{\sim}{L}}{\omega - i\nu - \omega_c} + \frac{\underset{\sim}{\parallel}}{\omega - i\nu} \right] \tag{8.7}$$

and the dispersion relation is

$$k_{\perp}^2 \left[1 - \frac{\omega_p^2 \, (\omega - i\nu)}{\omega[(\omega - i\nu)^2 - \omega_c^2]} \right] + k_{\parallel}^2 \left[1 - \frac{\omega_p^2}{\omega(\omega - i\nu)} \right] = 0 \ . \tag{8.8}$$

b) Maxwellian distribution: $F(\Lambda) = e^{-\frac{1}{2} \frac{T}{m} \Lambda^2}$.

In the tractable collisionless case, with $\theta = \omega_c u$,

$$\Lambda^2 = \frac{\underset{\sim}{k}}{\omega_c} \cdot \underset{\sim}{S}(\theta) \cdot \underset{\sim}{S}'(\theta) \cdot \frac{\underset{\sim}{k}}{\omega_c} = 2(k_{\perp}^2/\omega_c^2)(1 - \cos\theta) + (k_{\parallel}^2/\omega_c^2)\,\theta^2 \ . \tag{8.9}$$

Let

$$\lambda_{\perp} = \frac{k_{\perp}^2 T}{\omega_c^2 m} \, , \quad \lambda_{\parallel} = \frac{k_{\parallel}^2 T}{\omega_c^2 m} \; ; \quad \lambda = \lambda_{\perp} + \lambda_{\parallel} \ . \tag{8.10}$$

Then

$$F(\underset{\sim}{\Lambda}) = e^{-\lambda_\perp} e^{\lambda_\perp \cos\theta} e^{-\frac{1}{2}\lambda_\parallel \theta^2} , \tag{8.11}$$

so that

$$g_\perp = e^{-\lambda_\perp} \int_{-\infty}^{0} \sin\theta \, e^{i\Omega\theta} e^{\lambda_\perp \cos\theta} e^{-\frac{1}{2}\lambda_\parallel \theta^2} d\theta \tag{8.12}$$

and

$$g_\parallel = e^{-\lambda_\perp} \int_{-\infty}^{0} \theta \, e^{i\Omega\theta} e^{\lambda_\perp \cos\theta} e^{-\frac{1}{2}\lambda_\parallel \theta^2} d\theta , \tag{8.13}$$

where $\Omega = \omega/\omega_c$. By inspection of the integrands, it may be anticipated that the parallel component of $\underset{\sim}{k}$ will introduce collisionless damping.[13,7] The special case of perpendicular propagation is free of this and is of particular interest. The dispersion relation is then

$$1 = \frac{\omega_p^2}{\omega_c^2} e^{-\lambda} \int_{-\infty}^{0} \sin\theta \, e^{i\Omega\theta} e^{\lambda_\perp \cos\theta} d\theta \tag{8.14}$$

Integrating by parts and using the Fourier series

$$e^{\lambda\cos\theta} = \sum_{n=-\infty}^{\infty} I_n(\lambda) \, e^{in\theta} \tag{8.15}$$

yields

$$1 + \frac{k^2 T}{\omega_p^2 m} = e^{-\lambda} I_0(\lambda) + 2\Omega^2 \sum_{n=1}^{\infty} \frac{e^{-\lambda} I_n(\lambda)}{\Omega^2 - n^2} \; . \tag{8.16}$$

This dispersion relation for perpendicular propagation displays cyclotron harmonics. It is easily shown to agree with, but converge faster than, the version quoted by Stix.[14] It is equivalent to that given by Bernstein.[15]

c) Resonance distribution: $F(\underset{\sim}{\Lambda}) = e^{-v_1 |\underset{\sim}{\Lambda}|}$.

The collisionless case of perpendicular propagation is readily handled. From (8.9) with $k_{\parallel} = 0$,

$$F(\underset{\sim}{\Lambda}) = e^{-2\mu \left|\sin \theta/2\right|} \; , \tag{8.17}$$

where $\mu = v_1 k/\omega_c$. The dispersion relation is

$$1 = \frac{\omega_p^2}{\omega_c^2} \int_{-\infty}^{0} \sin\theta \, e^{i\Omega\theta} \, e^{-2\mu \left|\sin \frac{\theta}{2}\right|} d\theta$$

$$= \frac{\omega_p^2}{\omega_c^2} \sum_{n=0}^{\infty} e^{-i2\pi\Omega(n+1)} \int_0^{2\pi} e^{i\Omega\theta} \sin\theta \, e^{-2\mu \sin\frac{\theta}{2}} d\theta \; ,$$

which reduces to

$$1 = \frac{\omega_p^2}{\omega_c^2} \sum_{n=-\infty}^{\infty} \frac{(-1)^n I_{2n}(2\mu)}{(\Omega+n)^2 - 1} + \frac{\omega_p^2}{\omega_c^2} \sum_{n=-\infty}^{\infty} \frac{(-1)^{n+1} \cot\pi\Omega \; I_{2n+1}(2\mu)}{(\Omega + n + \frac{1}{2})^2 - 1} \; . \tag{8.18}$$

This also shows cyclotron harmonics but not, despite appearances, singular behavior when $\Omega = n + 1/2$.

d) Effects of drift and collisions: $F(\underset{\sim}{\Lambda}) = e^{i\underset{\sim}{\Lambda}\cdot\underset{\sim}{v}_0} F_0(\underset{\sim}{\Lambda})$.

A drifting distribution introduces no more than a Doppler shift into (8.1), as is physically evident from a transformation to a moving reference frame. If, however, there is drift of one plasma component relative to another, as exemplified by (7.7), the mean drift velocity differs from that of either component. There are then introduced into the integrand in (8.1) factors such as $\exp i\underset{\sim}{k}\cdot\underset{\sim}{S}(u)\cdot(\underset{\sim}{v}_0 - \underset{\sim}{v}_b)$, besides the Doppler shift. A brief discussion of their effect is in order.

By (4.15), the drift factors are of the form

$$\exp i\underset{\sim}{k}\cdot\underset{\sim}{S}(u)\cdot(\underset{\sim}{v}_0 - \underset{\sim}{v}_b) = \exp i\left[\frac{\underset{\sim}{k}\cdot(\underset{\sim}{v}_0 - \underset{\sim}{v}_b)}{\nu}(1 - e^{-\nu u})\right] \tag{8.19}$$

In the collisionless case, this is $\exp i\underset{\sim}{k}\cdot(\underset{\sim}{v}_0 - \underset{\sim}{v}_b)u$, which merely reassigns to each drifting component its proper Doppler shift. In the presence of collisions, the effect of this factor is clarified by the interpretation of the original integral (4.16) as a superposition of the perturbations of the past $(u < 0)$, as propagated to the present $(u = 0)$. The factor (8.19) is an oscillatory function $e^{i\phi(u)}$, of instantaneous frequency $d\phi/du = \underset{\sim}{k}\cdot(\underset{\sim}{v}_0 - \underset{\sim}{v}_b)\, e^{-\nu u}$; see Fig. 2. In the sufficiently distant past, this is of so rapid variation as to erase all memory of earlier perturbations, as confirmed by the well-known Riemann-Lebesgue theorem.[16] This oblivion-producing aspect of collisions is seen to be incorporated mathematically in the model of collisions adopted herein.

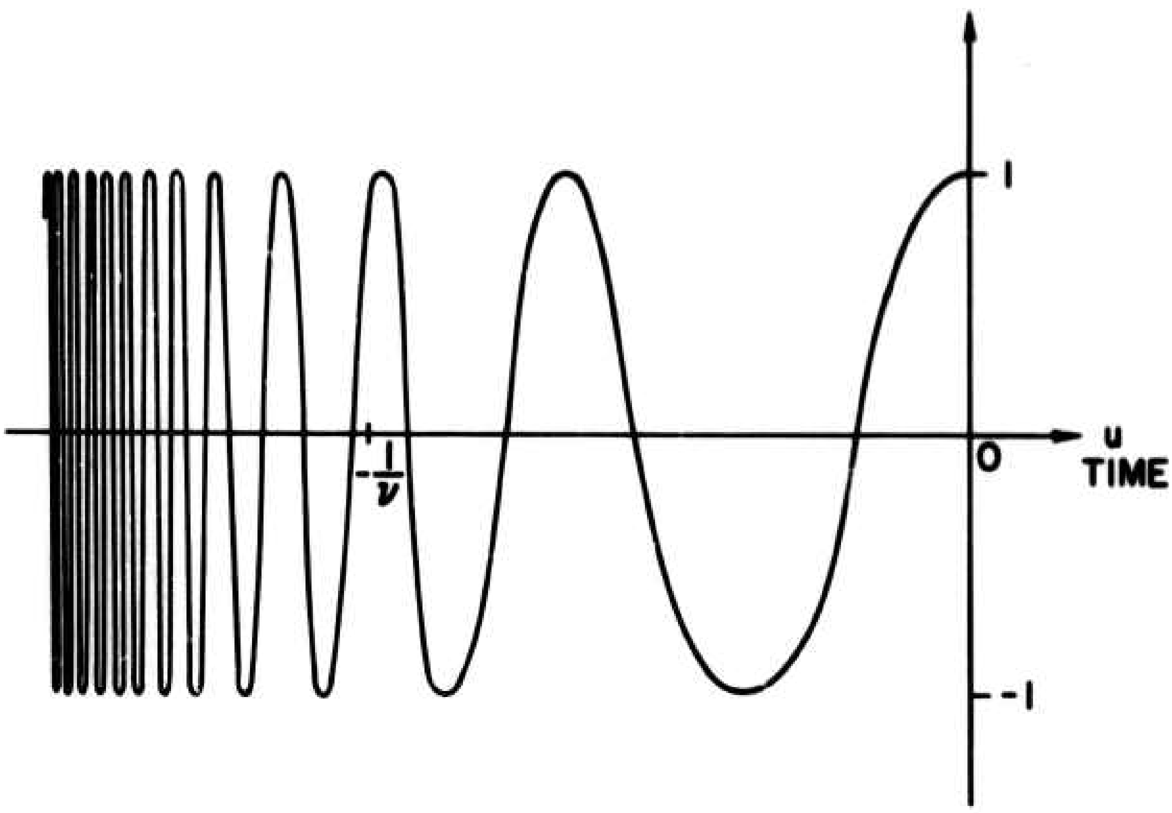

Figure 2 - The oblivion factor due to collisions

9. EXACT DISPERSION - B = 0

The exact dispersion relation, obtained from the full set of Maxwell's equations, is given in terms of the normalized conductivity tensor $\underset{\sim}{C}(\underset{\sim}{k},\omega)$ by (5.8). In this connection, it is useful to note the following properties of that equation. First, the electric field $\underset{\sim}{E}_1$ is to be an eigenvector corresponding to the prescribed eigenvalue. Second,

$$\mathrm{eig}\,(\alpha\underset{\sim}{A} + \beta\underset{\sim}{I}) = \alpha\,\mathrm{eig}\,\underset{\sim}{A} + \beta \tag{9.1}$$

is an identity. Third, if the conductivity tensor should take the form

$$\underset{\sim}{C} = \alpha\,\underset{\sim}{I} + \beta\,\underset{\sim}{kk}/k^2 \tag{9.2}$$

the dispersion relation (5.8) would become

$$\mathrm{eig}\left\{\left[\left(\frac{ck}{\omega}\right)^2 - \beta\right]\frac{\underset{\sim}{kk}}{k^2} - \alpha\right\} = \left(\frac{ck}{\omega}\right)^2 - 1 \tag{9.3}$$

or by (9.1),

$$1 = \alpha + \left(\frac{ck}{\omega}\right)^2 + \left[\beta - \left(\frac{ck}{\omega}\right)^2\right]\mathrm{eig}\,\frac{\underset{\sim}{kk}}{k^2} \tag{9.4}$$

But $\text{eig}\ \underset{\sim}{kk}/k^2 = 1$ for longitudinal modes, or zero for transverse modes. The dispersion relation then decomposes into

$$1 = \alpha + \beta \tag{9.5}$$

for longitudinal modes, and

$$1 = \alpha + \left(\frac{ck}{\omega}\right)^2 \tag{9.6}$$

for transverse modes. The matrix $\underset{\sim}{C}(\underset{\sim}{k},\omega)$ is given generally by (6.6-8). If $\underset{\sim}{C}$ reduces to α, a scalar, (9.5,6) still apply, but with $\beta = 0$.

In the absence of the external magnetic field, $\omega_c = 0$ and $\underset{\sim}{\Lambda} = (\underset{\sim}{k}/\nu)(e^{-\nu u} - 1)$ or $\underset{\sim}{\Lambda} = -\underset{\sim}{k}u$ when collisions are negligible. Then $\underset{\sim}{C}_e(\underset{\sim}{k},\omega)$ and $\underset{\sim}{C}_b(\underset{\sim}{k},\omega)$ reduce to

$$\underset{\sim}{C}_e(\underset{\sim}{k},\omega) = i\,\frac{\omega_p^2}{\omega} \int_{-\infty}^{0} e^{-i\underset{\sim}{\Lambda}\cdot\underset{\sim}{v}_0} \left(F + \frac{\partial F}{\partial \underset{\sim}{\Lambda}}\,\underset{\sim}{\Lambda}\right) e^{[\nu + i(\omega - \underset{\sim}{k}\cdot\underset{\sim}{v}_0)]u}\, du\ , \tag{9.7}$$

$$\underset{\sim}{C}_b(\underset{\sim}{k},\omega) = \frac{\omega_p^2}{\omega^2}\,\underset{\sim}{k} \times \int_{-\infty}^{0} e^{-i\underset{\sim}{\Lambda}\cdot\underset{\sim}{v}_0}\, \frac{\partial}{\partial \underset{\sim}{\Lambda}}\left(\frac{\partial F}{\partial \underset{\sim}{\Lambda}} \times \underset{\sim}{\Lambda}\right) e^{i(\omega - \underset{\sim}{k}\cdot\underset{\sim}{v}_0)u}\, du\ , \tag{9.8}$$

and $\underset{\sim}{C} = \underset{\sim}{C}_e + \underset{\sim}{C}_b$.

a) Cold plasma: $F(\underline{\Lambda}) = 1$.

Here, $\underline{C}_B = 0$ and

$$\underline{C}(\underline{k},\omega) = i\,\frac{\omega_p^2}{\omega}\int_{-\infty}^{0} e^{(\nu+i\omega)u}\,du = \frac{\omega_p^2}{\omega(\omega-i\nu)} \qquad (9.9)$$

which is a scalar. Hence, from (9.5,6), the dispersion relations are

$$1 = \frac{\omega_p^2}{\omega(\omega-i\nu)} \qquad (9.10)$$

for longitudinal modes, and

$$1 = \frac{\omega_p^2}{\omega(\omega-i\nu)} + \left(\frac{ck}{\omega}\right)^2 \qquad (9.11)$$

for transverse modes.

b) Maxwellian distribution: $F(\underline{\Lambda}) = e^{-\frac{1}{2}\frac{T}{m}\Lambda^2}$.

Here,

$$\frac{\partial F}{\partial \underline{\Lambda}} = -\frac{T}{m}\underline{\Lambda}\,F \ . \qquad (9.12)$$

so that, from (9.8), the r.f. magnetic field makes no contribution to $\underline{C}(\underline{k},\omega)$, which hence reduces to

$$\underline{C}(\underline{k},\omega) = i\,\frac{\omega_p^2}{\omega}\int_{-\infty}^{0}\left[\underline{1} - \frac{T}{m}\,\underline{\Lambda}\underline{\Lambda}\right] e^{(\nu+i\omega)u}\,e^{-\frac{1}{2}\frac{T}{m}\Lambda^2}\,du \ . \qquad (9.13)$$

In the collisionless case, this is

$$\underline{\varepsilon}(\underline{k},\omega) = 1 - i\,\frac{\omega_p^2}{\omega}\int_{-\infty}^{0}\left[\underline{1} - \left(\frac{Tk^2}{m\omega^2}\right)\left(\frac{\underline{k}\,\underline{k}}{k^2}\right)\theta^2\right] e^{i\theta}\, e^{-\frac{1}{2}\left(\frac{Tk^2}{m\omega^2}\right)\theta^2}\, d\theta \quad . \tag{9.14}$$

This has the form of (9.2); hence, the dispersion relations are

$$1 = i\,\frac{\omega_p^2}{\omega}\int_{-\infty}^{0}\left[1 - \left(\frac{Tk^2}{m\omega^2}\right)\theta^2\right] e^{i\theta}\, e^{-\frac{1}{2}\left(\frac{Tk^2}{m\omega^2}\right)\theta^2}\, d\theta \tag{9.15}$$

for longitudinal modes, and

$$1 = i\,\frac{\omega_p^2}{\omega}\int_{-\infty}^{0} e^{i\theta}\, e^{-\frac{1}{2}\left(\frac{Tk^2}{m\omega^2}\right)\theta^2}\, d\theta + \left(\frac{ck}{\omega}\right)^2 \tag{9.16}$$

for transverse modes. For comparison with the results of the quasistatic approximation, these may be simplified, by setting $y = \omega/k(2T/m)^{1/2}$, to

$$1 = \frac{\omega_p^2 m}{k^2 T}\,\frac{1}{y}\int_{-\infty}^{0}(2\xi^2 - 1)\, e^{-\xi^2 + 2y\xi}\, d\xi \tag{9.17}$$

for longitudinal waves, and

$$1 = \left(\frac{ck}{\omega}\right)^2 - \frac{\omega_p^2 m}{k^2 T}\,\frac{1}{y}\int_{-\infty}^{0} e^{-\xi^2 + 2y\xi}\, d\xi \tag{9.18}$$

for transverse waves. Letting, as before,

$$Z(y) = \int_{-\infty}^{0} e^{-\xi^2 + 2iy\xi}\, d\xi = \int_{0}^{\infty} e^{-\xi^2 - 2iy\xi}\, d\xi = e^{-y^2} \int_{iy}^{\infty} e^{-t^2}\, dt \qquad (9.19)$$

permits these to be written, after an integration by parts in (9.17),

for longitudinal waves,

$$1 = \frac{\omega_p^2 m}{k^2 T} \frac{dZ}{dy} \qquad (9.20)$$

and

$$1 = \left(\frac{ck}{\omega}\right)^2 - \left(\frac{\omega_p^2 m}{k^2 T}\right) \frac{1}{y} Z(y) \qquad (9.21)$$

for transverse waves. The exact result (9.20) for longitudinal modes is the same as (7.4) of the quasistatic approximation, which allows only longitudinal waves. Since $Z(y) \to 1/2y$ as $y \to \infty$, the low temperature limits of (9.20,21) are indeed the cold plasma relations (7.10,11) in the collisionless limit. In fact, the asymptotic expansion of $Z(y)$ for low temperature which, as is known,[17] does not reveal Landau damping, gives

$$1 = \frac{\omega_p^2}{\omega^2} \left(1 + 3 \frac{Tk^2}{m\omega^2}\right) \qquad (9.22)$$

[illegible]

$$[illegible] \qquad (9.23)$$

[illegible]

c) [illegible] distribution: $f(\underset{\sim}{\Lambda}) = e^{-v_1|\underset{\sim}{\Lambda}|}$.

[illegible]

$$\frac{\partial f}{\partial \underset{\sim}{\Lambda}} = -v_1 \frac{\underset{\sim}{\Lambda}}{|\underset{\sim}{\Lambda}|} f \qquad (9.24)$$

so that, again, $\underset{\sim}{C}_p(\underset{\sim}{k},\omega) = 0$, leaving only the r.f. electric field contribution. Hence,

$$\underset{\sim}{C}(\underset{\sim}{k},\omega) = i \frac{\omega_p^2}{\omega} \int_{-\infty}^{0} \left[\underset{\sim}{I} - v_1|\underset{\sim}{\Lambda}| \frac{\underset{\sim}{\Lambda}\underset{\sim}{\Lambda}}{\Lambda^2} \right] e^{(\nu+i\omega)u} \, e^{-v_1|\underset{\sim}{\Lambda}|} du \quad . \qquad (9.25)$$

In the collisionless case $\underset{\sim}{\Lambda} = -\underset{\sim}{k}u$ and $v_1|\underset{\sim}{\Lambda}| = -v_1 ku$, so that

$$\underset{\sim}{C}(\underset{\sim}{k},\omega) = i \frac{\omega_p^2}{\omega^2} \int_{-\infty}^{0} \left[\underset{\sim}{I} + \left(\frac{v_1 k}{\omega} \right) \frac{\underset{\sim}{k}\underset{\sim}{k}}{k^2} \theta \right] e^{i\theta} \, e^{(v_1 k/\omega)\theta} \, d\theta \quad , \qquad (9.26)$$

which is of the form (9.2). The dispersion relations are therefore

$$1 = i \frac{\omega_p^2}{\omega^2} \int_{-\infty}^{0} \left[1 + (v_1 k/\omega)\, \theta \right] e^{[i + (v_1 k/\omega)]\theta} \, d\theta \tag{9.27}$$

for longitudinal waves, and

$$1 = i \frac{\omega_p^2}{\omega^2} \int_{-\infty}^{0} e^{[i + (v_1 k/\omega)\, \theta]} d\theta + \left(\frac{ck}{\omega} \right)^2 \tag{9.28}$$

for transverse waves. Explicitly, these are, respectively,

$$1 = \frac{\omega_p^2}{(\omega - i v_1 k)^2} \tag{9.29}$$

and

$$1 = \frac{\omega_p^2}{\omega(\omega - i v_1 k)} + \left(\frac{ck}{\omega} \right)^2 , \tag{9.30}$$

which exhibit Landau damping.

d) Cold beam: $F(\underset{\sim}{\Lambda}) = e^{i \underset{\sim}{\Lambda} \cdot \underset{\sim}{v}_0}$.

Here, $\frac{\partial F}{\partial \underset{\sim}{\Lambda}} = i \, \underset{\sim}{v}_0 \, F$ and both the r.f. electric and magnetic fields contribute to $\underset{\sim}{C}$, yielding in the collisionless case,

$$\underset{\sim}{C}(\underset{\sim}{k}, \omega) = \frac{\omega_p^2}{\omega^2} \left(\underset{\sim}{I} + \frac{\underset{\sim}{v}_0 \underset{\sim}{k}}{\omega - \underset{\sim}{k} \cdot \underset{\sim}{v}_0} \right) \cdot \left(\underset{\sim}{I} + \frac{\underset{\sim}{k} \underset{\sim}{v}_0}{\omega - \underset{\sim}{k} \cdot \underset{\sim}{v}_0} \right) . \tag{9.31}$$

10. EXACT DISPERSION - $B \neq 0$

In the presence of a magnetic field, the exact expression for the normalized conductivity tensor entering the dispersion relation is, as previously given, $\underset{\sim}{C}(\underset{\sim}{k},\omega) = \underset{\sim}{C}_e(\underset{\sim}{k},\omega) + \underset{\sim}{C}_b(\underset{\sim}{k},\omega)$, where

$$\underset{\sim}{C}_e(\underset{\sim}{k},\omega) = i\,\frac{\omega_p^2}{\omega} \int_{-\infty}^{0} e^{-i\underset{\sim}{\Lambda}\cdot\underset{\sim}{v}_0} \left(F + \frac{\partial F}{\partial \underset{\sim}{\Lambda}}\underset{\sim}{\Lambda}\right) \cdot e^{[\underset{\sim}{Y} + i(\omega - \underset{\sim}{k}\cdot\underset{\sim}{v}_0)]u}\, du \ , \tag{10.1}$$

with $\underset{\sim}{Y} = \omega_c \underset{\sim}{X} + \nu$, and

$$\underset{\sim}{C}_b(\underset{\sim}{k},\omega) = \frac{\omega_p^2}{\omega^2}\, \underset{\sim}{k} \times \int_{-\infty}^{0} e^{-i\underset{\sim}{\Lambda}\cdot\underset{\sim}{v}_0} \frac{\partial}{\partial \underset{\sim}{\Lambda}} \left(e^{-\omega_c \underset{\sim}{X}u} \cdot \frac{\partial F}{\partial \underset{\sim}{\Lambda}} \times \underset{\sim}{\Lambda}\, e^{\omega_c \underset{\sim}{X}u}\right) e^{i(\omega - \underset{\sim}{k}\cdot\underset{\sim}{v}_0)u}\, du \ . \tag{10.2}$$

In these,

$$\underset{\sim}{\Lambda}(u) = -\underset{\sim}{k}\cdot\underset{\sim}{S}(u) = \underset{\sim}{k}\cdot\underset{\sim}{Y}^{-1}\left(e^{-\underset{\sim}{Y}u} - 1\right) \ . \tag{10.3}$$

The dispersion relation is

$$\mathrm{eig}\left(\frac{c^2}{\omega^2}\, \underset{\sim}{k}\,\underset{\sim}{k} - \underset{\sim}{C}\right) = \left(\frac{ck}{\omega}\right)^2 - 1 \ , \tag{10.4}$$

with the electric field as eigenvector.

a) Cold plasma: $F(\underset{\sim}{\Lambda}) = 1$.

$$\underset{\sim}{C}(\underset{\sim}{k},\omega) = i \frac{\omega_p^2}{\omega} \int_{-\infty}^{0} e^{(\omega_c \underset{\sim}{X} + \nu + i\omega)u} du$$

$$= \frac{\omega_p^2}{\omega} \left(\omega - i\nu - i\omega_c \underset{\sim}{X}\right)^{-1}$$

$$= \frac{\omega_p^2}{\omega} \left[\frac{\underset{\sim}{R}}{\omega + \omega_c - i\nu} + \frac{\underset{\sim}{L}}{\omega - \omega_c - i\nu} + \frac{\underset{\sim}{\parallel}}{\omega - i\nu} \right] . \tag{10.5}$$

The electric field vector must satisfy

$$\frac{c^2}{\omega^2} \underset{\sim}{k}\,\underset{\sim}{k}\cdot\underset{\sim}{E} - \frac{\omega_p^2}{\omega} \left(\omega - i\nu - i\omega_c \underset{\sim}{X}\right)^{-1} \cdot \underset{\sim}{E} = \left(\frac{c^2k^2}{\omega^2} - 1\right) \underset{\sim}{E}$$

or

$$c^2(\omega - i\nu)\, \underset{\sim}{k}\,\underset{\sim}{k}\cdot\underset{\sim}{E} - \left[\omega_p^2\, \omega + (c^2k^2 - \omega^2)(\omega - i\nu)\right] \underset{\sim}{E} = c^2 \omega_c \underset{\sim}{k}\cdot\underset{\sim}{E}\; i\underset{\sim}{X}\cdot\underset{\sim}{k} - (c^2k^2 - \omega^2)\omega_c\, i\underset{\sim}{X}\cdot\underset{\sim}{E} . \tag{10.6}$$

Waves of various linear and circular polarizations may be extracted from this equation.

b) Maxwellian distribution: $F(\underset{\sim}{\Lambda}) = e^{-\frac{1}{2}\frac{T}{m}\Lambda^2}$.

Since $\partial F/\partial \underset{\sim}{\Lambda} = -(T/m)\underset{\sim}{\Lambda} F$ and $\underset{\sim}{X}' = -\underset{\sim}{X}$, the cross product in $\underset{\sim}{C}_b(\underset{\sim}{k},\omega)$ vanishes, leaving

$$\underset{\sim}{C}(\underset{\sim}{k},\omega) = i\,\frac{\omega_p^2}{\omega}\int_{-\infty}^{0} F\,(\underset{\sim}{I} - \frac{T}{m}\underset{\sim}{\Lambda\Lambda})\, e^{(\omega_o \underset{\sim}{X}+\nu+i\omega)u}\, du \;. \tag{10.7}$$

c) Resonance distribution: $F(\underset{\sim}{\Lambda}) = e^{-v_1|\underset{\sim}{\Lambda}|}$.

Again, $\underset{\sim}{C}_b(\underset{\sim}{k},\omega) = 0$ and as $F + \frac{\partial F}{\partial \underset{\sim}{\Lambda}}\underset{\sim}{\Lambda} = (1 - \frac{v_1}{|\underset{\sim}{\Lambda}|}\underset{\sim}{\Lambda\Lambda})\,F$,

$$\underset{\sim}{C}(\underset{\sim}{k},\omega) = i\,\frac{\omega_p^2}{\omega}\int_{-\infty}^{0} F\,(\underset{\sim}{I} - \frac{v_1}{|\underset{\sim}{\Lambda}|}\underset{\sim}{\Lambda\Lambda})\, e^{(\omega_o \underset{\sim}{X}+\nu+i\omega)u}\, du \;. \tag{10.8}$$

In the cases of these last two distributions, only special cases of propagation and polarization parallel and perpendicular to the magnetic field can be considered tractable. It is useful to note the explicit expression

$$\Lambda^2 = e^{-\nu u}\left[k_\perp^{\,2}\,\frac{\sin^2\frac{\omega_c u}{2} + \sinh^2\frac{\nu u}{2}}{\left(\frac{\omega_c}{2}\right)^2 + \left(\frac{\nu}{2}\right)^2} + k_\parallel^{\,2}\,\frac{\sinh^2\frac{\nu u}{2}}{\left(\frac{\nu}{2}\right)^2}\right] \tag{10.9}$$

but the complexity of the quadratures precludes further development of general dispersion relations here. Of considerable interest, however, are the spatial and temporal decay rates of waveguide and cavity modes and the possibilities of amplification and oscillation when these can become negative.

11. TEMPORAL DECAY RATES

The rate of decay of energy in a cavity has been given in (5.17) as

$$2\alpha = n_0 q \int_{-\infty}^{0} \left\langle \frac{\partial}{\partial \underset{\sim}{v}} \cdot \left[e^{\underset{\sim}{\gamma} u} \cdot \underset{\sim}{R}(\underset{\sim}{s}, \underset{\sim}{w}, u) \cdot \underset{\sim}{v} \right] \right\rangle du \quad , \qquad (11.1)$$

where $\underset{\sim}{R}(\underset{\sim}{s}, \underset{\sim}{w}, u)$ is the correlation tensor

$$\underset{\sim}{R}(\underset{\sim}{s}, \underset{\sim}{w}, u) = \frac{1}{W_{av} T} \int_T \int_V \underset{\sim}{a}_1(\underset{\sim}{r} + \underset{\sim}{s}, \underset{\sim}{w}, t + u) \, \underset{\sim}{E}(\underset{\sim}{r}, t) \, dV dt \quad , \qquad (11.2)$$

the integrations being over the cavity volume V and a time interval T. In this equation, the acceleration $\underset{\sim}{a}_1(\underset{\sim}{r}, \underset{\sim}{v}, t)$ is the Lorentz one, given in (4.17). In (11.1), $\underset{\sim}{s}$ and $\underset{\sim}{w}$ are

$$\underset{\sim}{w} = \underset{\sim}{v}_0 + e^{-\underset{\sim}{\gamma} u} \cdot (\underset{\sim}{v} - \underset{\sim}{v}_0) \quad , \qquad \underset{\sim}{s} = \underset{\sim}{v}_0 u + (1 - e^{-\underset{\sim}{\gamma} u}) \, \underset{\sim}{\gamma}^{-1} \cdot (\underset{\sim}{v} - \underset{\sim}{v}_0) \quad , \qquad (11.3)$$

where $\underset{\sim}{\gamma} = \omega_c \underset{\sim}{X} + \nu$.

To evaluate the decay rate of a cavity mode, sufficient accuracy is obtainable by using the unperturbed field pattern in these expressions. The correlation tensor may be evaluated for any desired cavity mode and the substitution for $\underset{\sim}{w}$ and $\underset{\sim}{s}$ in (11.3) then permits the integration of (11.1) explicitly.

When the cavity mode is readily decomposed into plane waves, the calculation of the decay rates is particularly simple, as then the distributions in inverse velocity space are directly utilizable. The

calculations will here be illustrated for the simple case of the "cavity" consisting of infinite space, wherein a suitable mode is the plane wave $\underset{\sim}{E} = \text{Re}\, \underset{\sim}{E}_1 e^{i(\omega t - \underset{\sim}{k} \cdot \underset{\sim}{r})}$. The acceleration is then

$$\underset{\sim}{a}_1(\underset{\sim}{r}, \underset{\sim}{v}, t) = \text{Re}\, \frac{q}{m} \left[(1 - \frac{\underset{\sim}{k}}{\omega} \cdot \underset{\sim}{v}) \underset{\sim}{I} + \frac{\underset{\sim}{k}}{\omega} \underset{\sim}{v} \right] \cdot \underset{\sim}{E}_1 \, e^{i(\omega t - \underset{\sim}{k} \cdot \underset{\sim}{r})} \tag{11.4}$$

and the correlation tensor is readily found to be

$$\underset{\sim}{R}(\underset{\sim}{s}, \underset{\sim}{w}, u) = \frac{q}{m\epsilon_0} \text{Re} \left[(1 - \frac{\underset{\sim}{k}}{\omega} \cdot \underset{\sim}{w}) \underset{\sim}{I} + \frac{\underset{\sim}{k}}{\omega} \underset{\sim}{w} \right] \cdot \frac{\underset{\sim}{E}_1 \underset{\sim}{E}_1^*}{\underset{\sim}{E}_1 \cdot \underset{\sim}{E}_1^*} \, e^{i(\omega u - \underset{\sim}{k} \cdot \underset{\sim}{s})} . \tag{11.5}$$

Hence, by comparison with (5.11),

$$2\alpha = \text{Re} \left[-i\omega \frac{\underset{\sim}{E}_1^* \cdot \underset{\sim}{C}(\underset{\sim}{k},\omega) \cdot \underset{\sim}{E}_1}{\underset{\sim}{E}_1^* \cdot \underset{\sim}{E}_1} \right] = \text{Im}\, \frac{\underset{\sim}{E}_1^* \cdot \omega \underset{\sim}{C}(\underset{\sim}{k},\omega) \cdot \underset{\sim}{E}_1}{\underset{\sim}{E}_1^* \cdot \underset{\sim}{E}_1} . \tag{11.6}$$

The results previously obtained for $\underset{\sim}{C}(\underset{\sim}{k},\omega)$ may therefore be used directly to calculate decay rates in the plane wave case.

In the absence of the magnetic field, the decay rate for a cold plasma is, from (9.9)

$$2\alpha = \text{Im}\, \frac{\omega_p^2}{\omega - i\nu} = \frac{\omega_p^2 \nu}{\omega_p^2 + \nu^2} \tag{11.7}$$

and is, of course, due to collisions.

For a Maxwellian distribution, the decay rate obtained from (9.14) is, for longitudinal waves,

$$2\alpha = \mathrm{Re}\,\frac{\omega_p^2}{\omega}\int_{-\infty}^{0}(1-\gamma^2\theta^2)\,e^{i\theta}\,e^{-\frac{1}{2}\gamma^2\theta^2}\,d\theta$$

$$= \frac{\omega_p^2}{\omega}\int_{-\infty}^{0}(1-\gamma^2\theta^2)\cos\theta\,e^{-\frac{1}{2}\gamma^2\theta^2}\,d\theta = \frac{\omega_p^2}{\omega}\left(\frac{\pi}{2}\right)^{1/2}\frac{e^{-1/(2\gamma^2)}}{\gamma^3}\,, \tag{11.8}$$

which is the collisionless Landau damping decrement, with $\gamma^2 = Tk^2/m\omega^2$. For transverse waves, similarly,

$$2\alpha = \frac{\omega_p^2}{\omega}\int_{-\infty}^{0}\cos\theta\,e^{-\frac{1}{2}\gamma^2\theta^2}\,d\theta = \frac{\omega_p^2}{\omega}\left(\frac{\pi}{2}\right)^{1/2}\frac{e^{-1/(2\gamma^2)}}{\gamma}\,. \tag{11.9}$$

For the collisionless case of a resonance distribution, the corresponding decay rates are, from (9.29,30),

$$2\alpha = \mathrm{Im}\,\frac{\omega\,\omega_p^{\,2}}{(\omega - iv_1k)^2} = \frac{2\omega_p^{\,2}\,\omega^2\,v_1k}{(\omega^2 + v_1^{\,2}k^2)^2} \tag{11.10}$$

for longitudinal waves, and

$$2\alpha = \mathrm{Im}\,\frac{\omega_p^{\,2}}{\omega - iv_1k} = \frac{\omega_p^{\,2}\,v_1k}{\omega^2 + v_1^{\,2}\,k^2} \tag{11.11}$$

for transverse waves,

In the presence of the magnetic field, the decay rate for a cold plasma is, from (10.5),

$$2\alpha = \operatorname{Im} \omega_p^2 \, \hat{\underset{\sim}{e}}^* \cdot (\omega - i\nu - i\omega_c \underset{\sim}{X})^{-1} \cdot \hat{\underset{\sim}{e}}$$

$$= \omega_p^2 \nu \left[\frac{\hat{\underset{\sim}{e}}^* \cdot \underset{\sim}{R} \cdot \hat{\underset{\sim}{e}}}{(\omega+\omega_c)^2 + \nu^2} + \frac{\hat{\underset{\sim}{e}}^* \cdot \underset{\sim}{L} \cdot \hat{\underset{\sim}{e}}}{(\omega-\omega_c)^2 + \nu^2} + \frac{\hat{\underset{\sim}{e}}^* \cdot \underset{\sim}{\|} \cdot \hat{\underset{\sim}{e}}}{\omega^2 + \nu^2} \right] \tag{11.12}$$

where $\hat{\underset{\sim}{e}} = \underset{\sim}{E}_1 / |\underset{\sim}{E}_1|$. Note that $i\underset{\sim}{X}$, $\underset{\sim}{R}$, $\underset{\sim}{L}$, $\underset{\sim}{\|}$ are all hermitian matrices, whereas only the antihermitian part of $\underset{\sim}{C}(\underset{\sim}{k},\omega)$ contributes to the decay rate.

For a Maxwellian distribution, from (10.7),

$$2\alpha = \operatorname{Re} \omega_p^2 \, \hat{\underset{\sim}{e}}^* \cdot \int_{-\infty}^{0} (\underset{\sim}{I} - \frac{T}{m} \underset{\sim}{\Lambda\Lambda}) \, e^{(\omega_c \underset{\sim}{X} + \nu + i\omega)u} \, e^{-\frac{1}{2}\frac{T}{m}\Lambda^2} du \cdot \hat{\underset{\sim}{e}} \tag{11.13}$$

This reduces, for example, in the collisionless case of a parallel-propagating transverse wave, to

$$2\alpha = \frac{\omega_p^2}{\omega_c} \hat{\underset{\sim}{e}}^* \cdot \int_{-\infty}^{0} \cos\left(\frac{\omega}{\omega_c} - i\underset{\sim}{X}\right)\theta \; e^{-\frac{1}{2}\left(\frac{T k^2}{m\omega_c^2}\right)\theta^2} d\theta \cdot \hat{\underset{\sim}{e}}$$

$$= \omega_p \left(\frac{\pi}{2} \frac{m\omega_p^2}{Tk^2}\right)^{1/2} \left[e^{-\frac{m(\omega+\omega_c)^2}{2Tk^2}} \hat{\underset{\sim}{e}}^* \cdot \underset{\sim}{R} \cdot \hat{\underset{\sim}{e}} + e^{-\frac{m(\omega-\omega_c)^2}{2Tk^2}} \hat{\underset{\sim}{e}}^* \cdot \underset{\sim}{L} \cdot \hat{\underset{\sim}{e}} \right] .$$

(11.14)

The right- and left-handed circularly polarized waves are seen to suffer different Landau damping decrements.

The resonance distribution case analogous to this last one yields similarly

$$2\alpha = \mathrm{Re}\ \omega_p^2\ \hat{\underset{\sim}{e}}^* \cdot \int_{-\infty}^{0} e^{(\omega_c \underset{\sim}{X} + i\omega + v_1 k)u}\, du \cdot \hat{\underset{\sim}{e}}$$

$$= \omega_p^2\ v_1 k \left[\frac{\hat{\underset{\sim}{e}}^* \cdot \underset{\sim}{R} \cdot \hat{\underset{\sim}{e}}}{(\omega+\omega_c)^2 + v_1^2 k^2} + \frac{\hat{\underset{\sim}{e}}^* \cdot \underset{\sim}{L} \cdot \hat{\underset{\sim}{e}}}{(\omega-\omega_c)^2 + v_1^2 k^2} \right] \qquad (11.15)$$

The effects of a beam component of a plasma in a magnetic field are of special interest, because of the possibility of negative decay rates. An indication of how this can come about is provided by the following considerations. The inverse velocity space distribution function for a warm beam has the form $e^{i\underset{\sim}{\Lambda}\cdot\underset{\sim}{v}_0} F_0(\underset{\sim}{\Lambda})$, where $\underset{\sim}{v}_0$ is the beam velocity and $F_0(\underset{\sim}{\Lambda})$ gives its stationary distribution. In the expression for $\underset{\sim}{C}_e(\underset{\sim}{k},\omega)$, eq. (10.1), $F + (\partial F/\partial\underset{\sim}{\Lambda})\underset{\sim}{\Lambda}$ is hence replaced by $e^{i\underset{\sim}{\Lambda}\cdot\underset{\sim}{v}_0}\left[F_0 + (\partial F_0/\partial\underset{\sim}{\Lambda})\underset{\sim}{\Lambda} + i\underset{\sim}{v}_0\,\underset{\sim}{\Lambda}\,F_0\right]$. The exponential factor cancels in the integrand, leaving a Doppler shift. The last term, arising from the drift, can provide an imaginary component for $\underset{\sim}{C}_e(\underset{\sim}{k},\omega)$, whose sign depends on that of the drift velocity $\underset{\sim}{v}_0$ with respect to the direction of propagation. For example, the contribution to $\underset{\sim}{C}_e(\underset{\sim}{k},\omega)$ of this last term in the case of a beam with a Maxwellian distribution drifting along the external magnetic field is, in the absence of collisions,

$$\underset{\sim}{C}_{e0}(\underset{\sim}{k},\omega) = i\frac{\omega_p^2}{\omega}\int_{-\infty}^{0} i\underset{\sim}{v}_0\,\underset{\sim}{\Lambda}\,F_0(\underset{\sim}{\Lambda})\,e^{\left[\omega_c \underset{\sim}{X}+i(\omega-\underset{\sim}{k}\cdot\underset{\sim}{v}_0)\right]u}\,du \quad , \qquad (11.16)$$

so that for a parallel-propagating longitudinal wave, for which $\underset{\sim}{\Lambda}$ reduces to $-\underset{\sim}{k}u$, the contribution to the decay rate is, from (11.6),

$$2\alpha_{e0} = \mathrm{Im}\,\omega_p^2\,\underset{\sim}{\hat{b}}\cdot\underset{\sim}{v}_0\underset{\sim}{k}\cdot\underset{\sim}{\hat{b}}\int_{-\infty}^{0} u e^{-\frac{1}{2}\frac{T}{m}k^2u^2}\,e^{i(\omega-\underset{\sim}{k}\cdot\underset{\sim}{v}_0)u}\,du$$

$$= \omega_p^2\,v_0 k\int_{-\infty}^{0} u\,e^{-\frac{1}{2}\frac{T}{m}k^2u^2}\sin(\omega-\underset{\sim}{k}\cdot\underset{\sim}{v}_0)u\,du$$

$$= \omega_p^2\,v_0 k\left(\frac{\pi}{2}\right)^{1/2}\frac{\omega-\underset{\sim}{k}\cdot\underset{\sim}{v}_0}{\left[(T/m)k^2\right]^{3/2}}\,e^{-\frac{1}{2}\frac{(\omega-\underset{\sim}{k}\cdot\underset{\sim}{v}_0)^2}{(T/m)k^2}} \quad . \qquad (11.17)$$

This can be negative, either if $v_0 k < 0$, i.e. for upstream propagation, or if $\underset{\sim}{k}\cdot\underset{\sim}{v}_0 > \omega$, i.e. if the Doppler shifted frequency is negative. This negative Landau damping decrement would have to overcome the normal Landau damping provided by the other terms in the expression in order to leave net growth.

The Fourier analysis into plane waves is, of course, not necessary for the calculation of the decay rate for a cavity. The correlation tensor $\underset{\sim}{R}(\underset{\sim}{s},\underset{\sim}{w},u)$ is a property of the cavity field pattern which may be calculated separately before it is introduced into eq. (11.1).

12. SPATIAL DECAY RATES

The rate of decay of power flow along a waveguide may be calculated in a manner completely analogous to that of temporal decay rates. The formula is the same,

$$2\alpha = n_0 q \int_{-\infty}^{0} \left\langle \frac{\partial}{\partial \underset{\sim}{v}} \cdot \left[e^{\underset{\sim}{\gamma} u} \cdot \underset{\sim}{R}(\underset{\sim}{s}, \underset{\sim}{w}, u) \cdot \underset{\sim}{v} \right] \right\rangle du \quad , \tag{12.1}$$

but the correlation tensor is now

$$\underset{\sim}{R}(\underset{\sim}{s}, \underset{\sim}{w}, u) = \frac{1}{P_{av} T} \int_T \int_A \underset{\sim}{a}_1(\underset{\sim}{r}+\underset{\sim}{s}, \underset{\sim}{w}, t+u) \, \underset{\sim}{E}(\underset{\sim}{r}, t) \, dA \, dt \quad , \tag{12.2}$$

where the integrations are over the waveguide cross section A and a time interval T. In (12.1), $\underset{\sim}{s}$ and $\underset{\sim}{w}$ are functions of $\underset{\sim}{v}$ and u, as in (11.3). The correlation tensor is a property of the waveguide field pattern and may be calculated independently, before the substitutions for $\underset{\sim}{s}$ and $\underset{\sim}{w}$ make it a function of $\underset{\sim}{v}$ and u.

Again, the decomposition of the field pattern into plane waves, when convenient, expedites the calculations, and again the simple illustration of an infinite-space "waveguide" is instructive. For a plane wave, the energy density and power flow are related by the group velocity of the wave. Hence, the spatial decay rates for infinite plane waves are obtainable from the temporal decay rates derived above by dividing in each case by the undamped wave group velocity $c^2 k/\omega$.

For a rectangular waveguide supporting the TE_{10} mode

$\mathbf{E}(\mathbf{r},t) = \mathrm{Re}\,\hat{\mathbf{e}}\,\sin(\mathbf{k}\cdot\mathbf{r})e^{i(\omega t-\boldsymbol{\beta}\cdot\mathbf{r})}$ propagating along $\boldsymbol{\beta}$, polarized along $\hat{\mathbf{e}}$, with $\mathbf{k}\cdot\boldsymbol{\beta} = 0$ and $|\mathbf{k}| = \pi/a$, the correlation tensor is found to be

$$\mathbf{R}(\mathbf{s},\,\mathbf{w},\,u) = (q/mc^2\epsilon_0\beta)\ \mathrm{Re}\left[\mathbf{g}\cos\mathbf{k}\cdot\mathbf{s} + i\,\mathbf{h}\sin\mathbf{k}\cdot\mathbf{s}\right]\hat{\mathbf{e}}\ e^{i(\omega u-\boldsymbol{\beta}\cdot\mathbf{s})}\ , \tag{12.3}$$

where

$$\mathbf{g} = \hat{\mathbf{e}}(\omega-\boldsymbol{\beta}\cdot\mathbf{w}) + \boldsymbol{\beta}\,\hat{\mathbf{e}}\cdot\mathbf{w}\ , \qquad \mathbf{h} = \hat{\mathbf{e}}\,\mathbf{k}\cdot\mathbf{w} - \mathbf{k}\,\hat{\mathbf{e}}\cdot\mathbf{w}\ . \tag{12.4}$$

Hence, the spatial decay rate is obtainable as

$$2\alpha = (\omega_p^2/c^2\beta)\ \mathrm{Re}\ \hat{\mathbf{e}}\cdot\int_{-\infty}^{0}\left\langle\left(\mathbf{1} + \mathbf{v}\frac{\partial}{\partial\mathbf{v}}\right)\cdot e^{(\mathbf{Y}+i\omega)u}\ e^{-i\boldsymbol{\beta}\cdot\mathbf{s}}\cdot\left[\mathbf{g}\cos(\mathbf{k}\cdot\mathbf{s})+i\mathbf{h}\sin(\mathbf{k}\cdot\mathbf{s})\right]\right\rangle du\ , \tag{12.5}$$

with $\partial\mathbf{w}/\partial\mathbf{v} = e^{-\mathbf{Y}u}$ and $\partial\mathbf{s}/\partial\mathbf{v} = \mathbf{Y}^{-1}(1 - e^{-\mathbf{Y}u})$. If the trigonometric terms were decomposed into exponentials, the previously derived plane wave results, in terms of the distribution in inverse velocity space, could be used directly, by appropriate composition.

13. OTHER EXTERNAL FORCES

The results obtained apply to a plasma subjected to external forces limited to a constant magnetic field, together with an equivalent frictional drag force to represent collisions phenomenologically. The approach can be readily generalized to allow more complicated force fields. These might include gravity, d.c. electric fields, time varying pump fields, and incident waves. The latter two would result in parametric effects and various wave-wave interactions.

To introduce any external force field requires only the solution of the dynamics problem giving the unperturbed orbit. For example, there may be added to the magnetizing field and viscous force a constant electric bias field, or gravity. Then

$$\underset{\sim}{a}_0 = -(\omega_c \underset{\sim}{X} + \nu)\underset{\sim}{v} + (q/m)\,\underset{\sim}{E}_0 + \nu \underset{\sim}{v}_0 \tag{13.1}$$

and the orbit is given by

$$\underset{\sim}{v}(\tau) = e^{-\underset{\sim}{Y}u} \cdot \underset{\sim}{v} - \underset{\sim}{Y}^{-1}\,(e^{-\underset{\sim}{Y}u} - 1)\left[(q/m)\underset{\sim}{E}_0 + \nu\,\underset{\sim}{v}_0\right] \tag{13.2}$$

and

$$\underset{\sim}{r}(\tau) = \underset{\sim}{r} + \underset{\sim}{Y}^{-1}\,(1 - e^{-\underset{\sim}{Y}u}) \cdot \underset{\sim}{v} - \left[\underset{\sim}{Y}^{-2}(1 - e^{-\underset{\sim}{Y}u}) - \underset{\sim}{Y}^{-1}u\right]\left[(q/m)\underset{\sim}{E}_0 + \nu \underset{\sim}{v}_0\right], \tag{13.3}$$

where $u = \tau - t$. These are to be substituted for $\underset{\sim}{w}$ and $\underset{\sim}{r} + \underset{\sim}{s}$,

respectively, in the various equations.

For a time varying applied electric field $E_0(t)$, the orbit is

$$\underset{\sim}{v}(\tau) = \underset{\sim}{v}_0 + e^{-\underset{\sim}{Y}u} \cdot (\underset{\sim}{v}-\underset{\sim}{v}_0) + (q/m) \int_0^u e^{-\underset{\sim}{Y}\xi} \underset{\sim}{E}_0(\tau - \xi)\, d\xi \quad . \qquad (13.4)$$

In particular, for a harmonic pump field $\underset{\sim}{E}_0(t) = \underset{\sim}{E}e^{i\omega_0 t}$, this reduces to

$$\underset{\sim}{v}(\tau) = \underset{\sim}{v}_0 + e^{-\underset{\sim}{Y}u} \cdot (\underset{\sim}{v}-\underset{\sim}{v}_0) + (\underset{\sim}{Y} + i\omega_0)^{-1}\left[e^{i\omega_0 u} - e^{-\underset{\sim}{Y}u}\right] (q/m)\underset{\sim}{E}e^{i\omega_0 t} \quad . \quad (13.5)$$

If the pump is an incident wave $\underset{\sim}{E}(\underset{\sim}{r},t) = \underset{\sim}{E}_0 e^{i(\omega_0 t-\underset{\sim}{k}_0\cdot\underset{\sim}{r})}$, the orbit is given by the solution to

$$\frac{d^2\underset{\sim}{r}}{d\tau^2} + \underset{\sim}{Y}\frac{d\underset{\sim}{r}}{d\tau} = \frac{q}{m}\left[\underset{\sim}{E}_0 + \frac{d\underset{\sim}{r}}{d\tau} \times \left(\frac{\underset{\sim}{k}_0}{\omega_0} \times \underset{\sim}{E}_0\right)\right] e^{i\left[\omega_0\tau - \underset{\sim}{k}_0\cdot\underset{\sim}{r}(\tau)\right]} + \nu\, \underset{\sim}{v}_0 \quad , \qquad (13.6)$$

with $\underset{\sim}{v}(\tau) = d\underset{\sim}{r}/d\tau$ and subject to $\underset{\sim}{r}(\tau) = \underset{\sim}{r}$ and $\underset{\sim}{v}(\tau) = \underset{\sim}{v}$ at $\tau = t$. Since the entire approach is perturbational, it would not be inconsistent to use orbits obtained from this equation by successive approximations or linearization.

14. CONCLUSIONS

A formalism has been presented which unifies the analysis of the various wave properties of a plasma subjected to some external force fields, particularly magnetization. The aim has been to derive relations of a general nature, into which there may be introduced the appropriate descriptions of both the unperturbed orbit in the applied field and the equilibrium velocity distribution of the plasma constituents. Straightforward quadratures then yield the wave dispersion relations, the permittivity or conductivity tensors, the charge or current distributions, and absorption or growth rates, both temporal and spatial, as well as the perturbed velocity distribution.

In summary, the Boltzmann equation has been linearized by separating the acceleration into externally applied and internally induced components, with collisions considered phenomenologically as a viscous retarding force. The solution to the linearized equation was taken beyond merely that for the perturbing velocity distribution to obtain directly an expression for the perturbation of any macroscopic quantity which can be calculated as an expectation value with respect to the unperturbed velocity distribution.

The effects of collisions have been accounted for in a convenient, yet not unrealistic, manner. With just an effective relaxation rate ν as parameter, they were considered as simply damping the otherwise helical unperturbed orbits of the constituent particles. Collisions appeared as an additional, effectively external, viscous drag force tending to relax the velocities toward the mean flow velocity. This model is mathematically

tractable and avoids the paradoxes associated with indiscriminate conversions of real frequencies to the complex form $\omega - i\nu$. It achieves consistency between quasistatic and exact results and reduces properly to the cold plasma fluid model. In the formulation interpreted as superpositions of perturbations along the particle trajectories, the collision model introduces factors that tend to destroy the system's memory of perturbations suffered in the distant past, as measured by ν^{-1} . There is thus provided a simple mathematical model of physical collisional effects.

The key quantities obtainable in this formalism as expectation values are the permittivity and conductivity tensors, from which the dispersion relation can be extracted by combination with either the Poisson equation or Maxwell's equations. Spatial and temporal decay or growth rates were expressed in terms of similar correlation tensors associated with the waveguide or cavity field pattern.

In the case of plane waves, it was found that most quantities of interest are expressible through the expectation value of an exponential function of velocity. This was the basis for a significant simplification introduced by expressing the equilibrium velocity distribution in inverse velocity space; i.e. as a Fourier transform in velocity. A variety of expectation values are then obtainable by simple differentiation or convolutions. In addition, singular complex integrations are thereby avoided and phenomena such as Landau and cyclotron damping appear naturally.

Explicit results have been presented under the quasistatic

approximation and without it, in the absence and in the presence of the applied magnetic field. Dispersion relations have been derived for these cases for cold plasmas, for Maxwellian plasmas, for resonance distributions, and for beams or drifting plasmas, in some cases with collisions included explicitly.

Temporal and spatial decay rates, particularly Landau and cyclotron damping, have been calculated for plane waves by simple integration. The quadratures necessary in less tractable cases have been indicated. Finally, the requirements for generalizing the theory to include parametric interactions have been presented.

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A method is presented for unifying the analysis of various wave properties of a plasma in a magnetic field. An expression is derived for any microscopic perturbation quantity as an integral of an expectation value with respect to the equilibrium distribution. This yields permittivity and conductivity tensors, and hence the dispersion relation, or spatial and temporal decay or growth rates, for any specified velocity distribution. In the plane wave case, the averaging is eliminated and the calculation significantly simplified by transformation to "inverse velocity space," so that singular integrals do not appear and phenomena such as Landau damping become evident. Quasistatic and exact dispersion relations are derived for cold, Maxwellian, resonance, and drifting distributions. Collisions are accounted for as a viscous drag force along the orbits. Generalizations to other external force fields are discussed.

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